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MATHEMATICS 31

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MODULES 1–8

Learning Facilitator's Manual

Distance Learning

Alberta EDUCATION

Mathematics 31

LEARNING FACILITATOR'S MANUAL

Distance Learning

Alberta EDUCATION

NOTE: This Mathematics 31 Learning Facilitator's Manual contains answers to teacher-assessed assignments and the final test; therefore, it should be kept secure by the teacher. Students should not have access to these assignments or the final test until they are assigned in a supervised situation. The answers should be stored securely by the teacher at all times.

ACKNOWLEDGEMENTS

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Printing: LRDC Production Division, Barrhead

This document is intended for	
Students	
Teachers (Mathematics 31)	✓
Administrators	
Parents	
General Public	
Other	

Mathematics 31
Learning Facilitator's Manual
Modules 1–8
Alberta Distance Learning Centre
ISBN 0-7741-1112-7

Teachers

Register with the Alberta Distance Learning Centre

The Alberta Distance Learning Centre is dedicated to upgrading and continually improving your Learning Facilitator's Manual so that it accurately reflects any necessary revisions we have had to make in the student module booklets, assignment booklets, or the sample final test. The types of revisions that will be made are those that make the course more accurate, current, or more effective.

The ADLC will send you the **latest enhancements or minor upgrades** for your Learning Facilitator's Manual if you return the following registration card to: Alberta Distance Learning Centre, Box 4000, Barrhead, Alberta T7N 1P4, Attention: Instructional Design and Development.

ADLC Learning Facilitator's Manual
Registration Card

First Name	Surname
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City	Postal Code
Course Title	Approximate Date of Purchase

You can help ensure that distance learning courseware is of top quality by letting us know of areas that need to be adjusted. Call the Alberta Distance Learning Centre free of charge by using the RITE line and ask for the Editing Unit. Also, a teacher questionnaire has been included at the back of most Learning Facilitator's Manuals. Please take a moment to fill this out.

We look forward to hearing from you!

Contents

Introduction

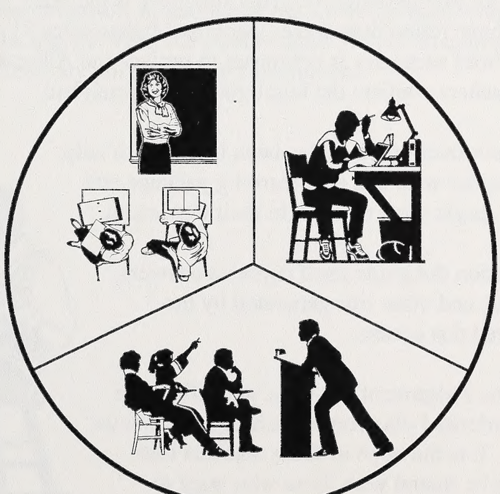
A survey of these course materials will confirm that this new learning package has been specially designed for many kinds of teachers working in a variety of situations.

In Which Category Do You Fit?

- ❑ Small Schools Teacher
  - ❑ inexperienced
  - ❑ experienced, but in other subject areas
  - ❑ experienced in teaching Mathematics 31, but wanting to try a different approach

- ❑ Distance Learning Teacher
  - ❑ travelling to schools within the jurisdiction
  - ❑ using facsimile and teleconferences to teach students within the area

- ❑ Large Schools Teacher
  - ❑ inexperienced
  - ❑ experienced in teaching Mathematics 31, but wanting to try a different approach



Because these materials have been created by experienced classroom teachers and distance learning specialists, they have many advantages for students and teachers regardless of their situations.

Advantages for Students

- incorporates a strong learner-centred philosophy
- promotes such qualities in the learner as autonomy, independence, and flexibility
- is developed through media which suit the needs and circumstances of the learner
- reflects the experiential background of Alberta students
- opens up opportunities by overcoming barriers that result from geographical location
- promotes individualized learning, allowing learners to work at their own pace

Advantages for Teachers

- allows teachers maximum teaching time and minimizes preparation time
- includes different routes through the materials to suit different learners
- incorporates a wide range of teaching strategies, in particular those using independent and individual learning
- delivers curriculum designed by education specialists that reflects the Alberta Education Program of Studies with an emphasis on Canadian content
- provides learning materials which are upwardly compatible with advanced educational technology

Does it sound like something you could use?

This Learning Facilitator's Manual begins with an overview of the current Alberta Education Program of Studies for Mathematics 31. This summary is included for inexperienced teachers or those teachers who have found themselves teaching Mathematics 31 when their training is in other subject areas. This brief summary is not meant to replace the Alberta Education Program of Studies, but rather to help teachers confirm the highlights of the program.

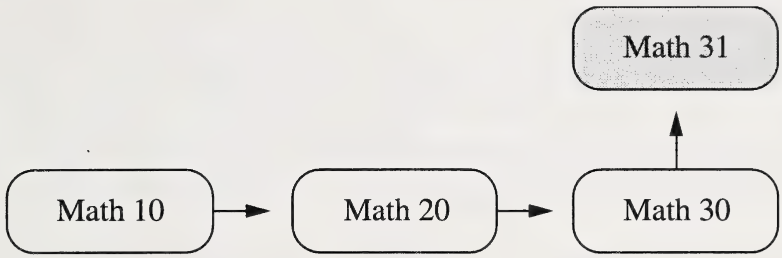
Other parts of this introduction have also been included to help teachers become familiar with this new learning package and determine how they might want to use it in their classroom.

Beyond the introduction the guide itself contains answers, models, explanations, and other tips generated by the teachers who authored this course.

The module booklets, assignment booklets, and LFMs are the products of experienced classroom teachers and distance learning specialists. It is the hope of these teachers that their experience can be shared with those who want to take advantage of it.

Overview of the Program of Studies

Mathematics 31 is the culminating course of the Mathematics 10–20–30 and Mathematics 31 course sequence. Mathematics 31 is normally taken after Mathematics 30; however, Mathematics 31 and Mathematics 30 may be taken concurrently.



Mathematics 10–20–30 and Mathematics 31 are designed for students who have achieved the acceptable standard in Mathematics 9, and who are intending to pursue studies beyond high school at a university or in a mathematics-intensive program at a technical school or college.

Having successfully achieved the acceptable standard in each of Mathematics 10, Mathematics 20, and Mathematics 30, students will have met or exceeded the basic entry requirements in mathematics for most university or university-transfer programs; and will have met or exceeded all entry requirements in mathematics for technical school or college programs. A significant number of students would benefit by taking a course in calculus that improves their opportunity for success in higher-level courses in mathematics at university, such as those required by mathematics and physics honours programs, or by engineering and business programs.

Mathematics 31 emphasizes the theoretical and practical development of topics in the algebra of functions, trigonometry, differential calculus, and integral calculus up to a standard acceptable for entry into all first-year programs in mathematics, science, engineering, and business. The course is designed to bridge the gap between the Mathematics 10–20–30 course sequence and the calculus course sequences offered by post-secondary institutions.

Overview of Mathematics 31

The Mathematics 31 course is designed to introduce students to the mathematical methods of differential and integral calculus. The course acts as a bridge between the outcomes of the Mathematics 10–20–30 program and the requirements of the mathematics encountered in post-secondary programs.

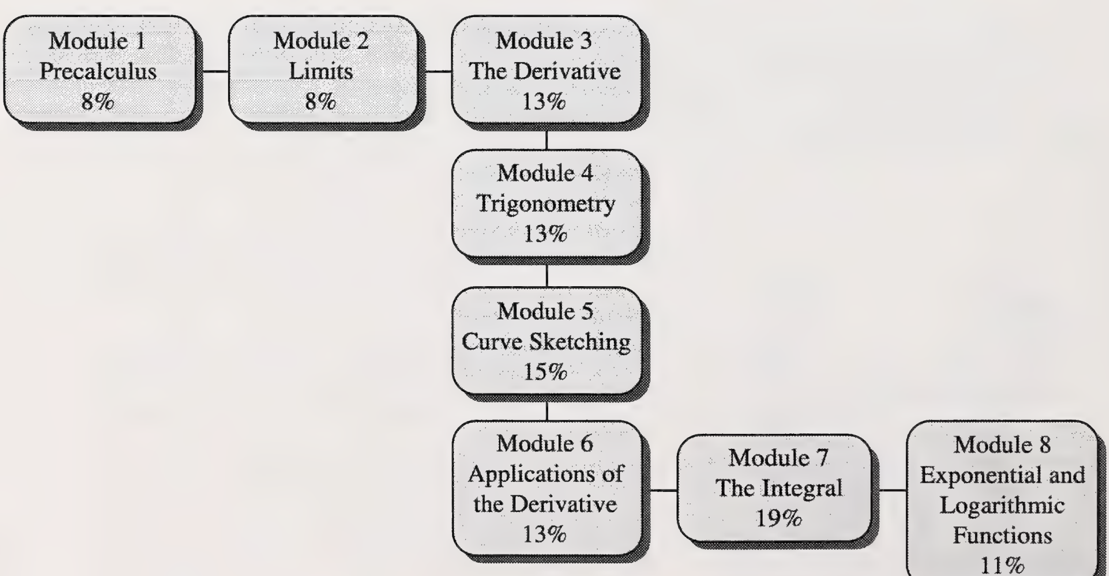
The broader focuses of Mathematics 31 are to develop a student's appreciation of the beauty and power of mathematics in problem solving and expression of human thought. As well, students are expected to create solutions to problems and to communicate those solutions. Also, a major part of mathematics is reasoning: to analyse; to conjecture; to draw conclusions and justify solutions; and to make connections between mathematical concepts, between mathematics and other subjects, and between mathematics and everyday life.

The course consists of eight modules. The first two modules build on the student's existing skills in working with functions and expands this knowledge to include the study of limits to prepare for a study of differential and integral calculus.

Modules 3 through 6 deal with the derivative and its applications to graphing and real-world applications for both algebraic and trigonometric functions.

Modules 7 and 8 deal with the integral and its applications. Module 8 focuses specifically on exponential and logarithmic functions.

Because of the sequential nature of the course, the student is expected to study the modules in order. The percentages given represent the amount of time that should be spent on each module.

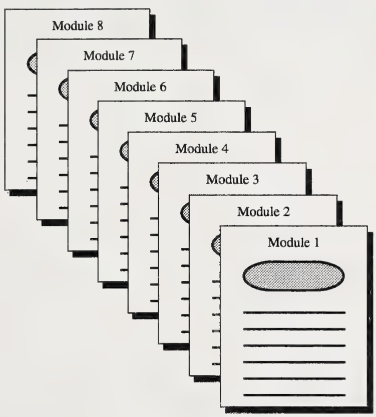


Structure of the Learning Package

Basic Design

This new learning package involves many other components in addition to the Learning Facilitator's Manual.

Modules



The print components involve many booklets called modules. These modules contain guided activities that instruct students in a relevant, realistic setting.

The modules have been specially designed to promote such qualities in the learner as autonomy, independence, and flexibility. Writers have incorporated such teaching strategies as working from the concrete to the abstract, linking the old to the new, getting students actively involved, and using advance, intermediate, and post organizers. Many other techniques enable learners to learn on their own for at least some of the time.

Contents
Overview Evaluation
Section 1 Activity 1 Activity 2 etc.
Section 2 Activity 1 Activity 2 etc.
Section 3 Activity 1 Activity 2 etc.
Section 4 Activity 1 Activity 2 etc.
Module Summary

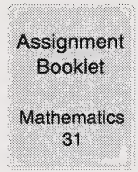
The structure of the module booklets follows a systematic design. Each module begins with a detailed table of contents which shows the students all the main steps. It acts as an organizer for students. The overview introduces the module topic or theme. A graphic representation has been included to help visual learners and poor readers. The introduction also states the weightings of each assignment.

The body of the module is made up of two or more closely related sections. Each section contains student activities that develop skills and knowledge centred around a theme.

The activities may involve print, audio, video, computer, or laser videodisc formats. At times the student and the learning facilitator are allowed to choose the activity that best suits the student's needs and interests. Other activities such as the Extra Help and Enrichment are optional pathways. This flexibility caters to each student's personal situation.

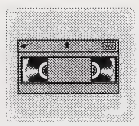
The summary focuses on the skills and strategies that the student has learned.

Assignment Booklet

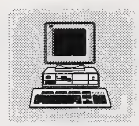


Accompanying each module is an assignment booklet. The activities in these booklets can be used for formative and for summative assessments. The students should complete these assignment booklets when they have thoroughly reviewed the module materials. The assignment booklets have been designed for classroom use, or for mailing. **If the booklets are not being mailed, you should remove the outside cover.**

Media



VIDEOCASSETTE



COMPUTER

The package also includes references to media. Some types of media such as computer disks and laser videodiscs are optional choices for students; however, there are activities that require students to view certain videos. These mandatory videos are listed on the following page. It is important that you acquire these videos as you are planning the course. In addition to the mandatory videos, optional videos have been mentioned at various points in the modules. A list of the optional videos is also included on the following page. More information about the videos can be found within the LFM.

A special audiocassette features a teacher guiding the student through the course. The appearance of the teacher icon reminds students that there is this additional help available. If the students are working individually, you may find this cassette a valuable asset. If you are working in a large group, you may wish to guide the students yourself.

Materials, Media, and Equipment

Mandatory Components

Equipment (Hardware)	Media	Materials
VCR	Mandatory Video List: • Mathematics 31, Video 1 Catch 31: – An Introduction to Calculus and Vectors – The Derivative of First Principles: The Power Rule – The Sum Rule: The Chain Rule – Product and Quotient Rule • Mathematics 31, Video 2 Catch 31: – Derivatives and Graph Sketching – Maxima and Minima – Distance, Velocity and Acceleration – Derivatives for Relations • Mathematics 31, Video 3 Catch 31: – Related Rates – Integration – Areas Under or Between Curves	• LFM for Mathematics 31 • one complete set of module booklets (8) and assignment booklets (8) for each student • There is a final test.

Videocassettes or laser videodiscs used in the course may be available from the Learning Resources Distributing Centre or ACCESS Network. You may also wish to call your regional library service for more information.

Using This Learning Package in the Classroom

Conventional Classroom

Whether your classroom has desks in rows or tables in small groups, you may be most comfortable with a learning system that you can use with all your students in a paced style. In other words, you may want a package that will suit all of your students, so they can move through the materials as one group or several small groups. Because these materials contain different routes or pathways within each module, they can address various learning styles and preferences. The materials also include many choices within the activities to cater to different thinking levels and ability levels. Because of their versatility and flexibility, these materials can easily suit a conventional classroom.

Open-Learning Classroom

Open learning is the concept of opening up opportunities by overcoming barriers of time, pace, and place by giving the learners a package specially designed to enable them to learn on their own for at least some of the time.

Such a concept is not new. Many teachers can recite attempts to establish an individualized learning system as they recognized the importance of trying to personalize courseware to meet each individual student's needs. But these efforts often failed due to lack of time and lack of quality materials that conformed to Alberta specifications.

Due to advanced educational technology and improved Alberta-specific learning packages, a student-centred approach is now possible. Improved technology now allows us to provide support to learners individually, regardless of their pace or location. A teacher cannot be in twenty-eight places at one time offering guidance. However, media and a well-designed learning package can satisfy individual needs. Technology can also help provide an effective management system needed to track the students as they progress independently through the materials.

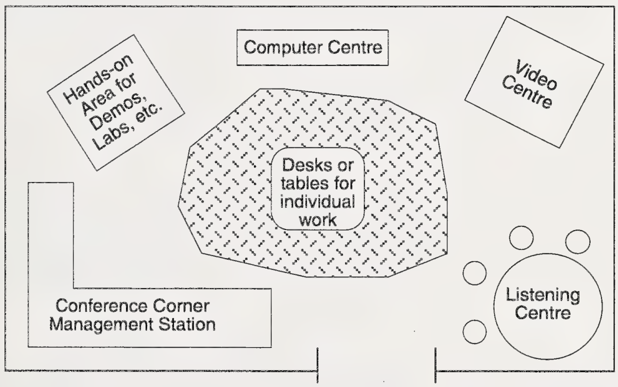
The key to a successful open-learning system depends on three vital elements: a learning package specially designed to enable students to learn effectively on their own for at least some of the time; various kinds of learner support; and a management system and style that ensures that the open-learning system runs smoothly.

The Key to a Successful Open-Learning System

Learning Package

The specially designed learning package needed for a successful open-learning system has been developed for you. The objectives teach current Alberta specifications using strategies designed for individualized instruction. As the learning facilitator, you need to be sure to have all the components in the learning package available to students as needed.

If adequate numbers of media are available to satisfy the demand, a centre can be established for specific media.



You may not have the luxury to have enough hardware to set up a permanent video or computer centre in your classroom. In that case, students should be encouraged to plan ahead. Perhaps every three to five days they should preview their materials and project when they would need a certain piece of media. This would allow you to group students, if necessary, or reserve media as required.

Support

Support is definitely a key element for successful learning, and when you're planning an individualized, non-paced program, you need to carefully plan when and how support will be given.

The materials contain a form of consistent support by providing immediate feedback for activities included in the module booklet. High school students have solutions, models, explanations, and guides included in the appendix of every module booklet. These are included so students can receive immediate feedback to clarify and reinforce their basic understanding before they move on to higher levels of thinking.

As the learning facilitator, you may be needed to offer more personal guidance to those students having difficulty, or you may need to reinforce the need for students to do these activities carefully before attempting the assignments in the assignment booklet.

The activities include choices and pathways. If a student is having difficulty, you may need to encourage that student to work on all the choices rather than one. This would provide additional instruction and practice in a variety of ways.

Another form of support is routine contact with each individual. This might be achieved with a biweekly conference scheduled by you, or as students reach a certain point (e.g., after each section is completed), they may be directed to come to the conference area.

Special counselling may be needed to help students through difficult stages. Praise and encouragement are important motivators, particularly for those students who are not used to working independently.

Direct teaching may be needed and scheduled at certain points in the program. This might involve small groups or a large group. It might be used to take advantage of something timely (e.g., election, eclipse, etc.), something prescheduled like the demonstration of a process, or something involving students in a hands-on, practical experience.

Support at a distance might include tutoring by phone, teleconferencing, faxing, or planned visits. These contacts are the lifeline between learners and distance education teachers, so a warm dialogue is essential.

Management

Good management of an open-learning system is essential to the success of the program. The following areas need action to ensure that the system runs smoothly:

- Scheduling, Distributing, and Managing Resources – As discussed earlier, this may require a need for centres or a system for students to project and reserve the necessary resources.

- Scheduling Students – Students and teachers should work together to establish goals, course completion timelines, and daily timelines. Although students may push to continue for long periods of time (e.g., all morning), teachers should discourage this. Concentration, retention, and motivation are improved by taking scheduled breaks.

- Monitoring Student Progress – You will need to record when modules are completed by each student. Your data might also include the projected date of completion if you are using a student contract approach.

Sample of a Student Progress Chart

Mathematics 31		Module 1	Module 2	Module 3	Module 4	Module 5	Module 6	Module 7	Module 8	Final Test
Billy Adams	P									
	A									
Louise Despins	P									
	A									
Violet Klaissian	P									
	A									
P = Projected Completion Date A = Actual Completion Date										

The student could keep a personal log as well. Such tracking of data could be stored easily on a computer.

- Recording Student Assessments – You will need to record the marks awarded to each student for work completed in each module assignment booklet. The marks from these assignment booklets will contribute to a portion of the student's final mark. Other criteria may also be added (a special project, effort, attitude, etc.). Whatever the criteria, they should be made clear to all students at the beginning.

Sample of a Student Assessment Chart

Mathematics 31	Module 1	Module 2	Module 3	Module 4	Module 5	Module 6	Module 7	Module 8	Year's Average	Final Test	Final Mark
Billy Adams	67	65	54	47	78	67	70	55	63		
Louise Despins	43	50	54	55	48	42	45	56	49		
Violet Klaissian	65	65	66	68	67	70	67	68	67		

Letter grading could easily be substituted.

- Recording Effectiveness of System – Keep ongoing records of how the system is working. This will help you in future planning.

Sample of a System Assessment Chart

Module 1			
Date	Module Booklet	Assignment Booklet	Resources/Media

The Role of the Teacher in an Open-Learning Classroom

The teachers in a conventional classroom spend a lot of time talking to large groups of learners. The situation in open learning requires a different emphasis. Teachers will probably meet learners individually or in very small groups.

With this approach it is necessary to move beyond the idea of a passive learner depending largely on a continually supportive teacher. The teacher must aim to build the student's confidence, to stimulate the learner into self-reliance, and to guide the learner to take advantage of routes that are most meaningful and applicable to the learner.

These materials are student-centred, not teacher-centred. The teacher needs to facilitate learning by providing general support to the learner.

Evaluation

Evaluation is important to the development of every learner. Data gathering and processing, and decision making, at the student and teacher level, serve as means of identifying strengths and weaknesses.

These specially designed learning packages contain many kinds of informal and formal evaluation.

Observation

In the classroom the teacher has the opportunity to see each student perform every day and to become aware of the level and nature of each student's performance.

Observations are more useful if they are recorded in an organized system. The following list of questions is a sample of types of observations and how they can be collected.

Observation Checklist

	B. Adams	L. Despins	V. Klaissian	H. Smith	K. Dalley
1. Does the student approach the work in a positive manner?					
2. Is the student struggling with the reading level?					
3. Does the student make good use of time?					
4. Does the student apply an appropriate study method?					
5. Can the student use references effectively, etc.?					

Observation may suggest a need for an individual interview with a student.

Individual Conferences

Individual conferences may be paced (scheduled) by the calendar, at certain points in the module, or they may be set up only as needed or requested.

During these conferences teachers can determine the student's progress and can assess the student's attitudes toward the subject, the program, school, and self, as well as the student's relationship with other students. With guided questions the teacher can encourage oral self-assessment; the student can discuss personal strengths or weaknesses in regard to the particular section, module, or subject area.

Self-Appraisal

Self-appraisal helps students recognize their own strengths and weaknesses. Through activities that require self-assessment, students also gain immediate feedback and clarification at early stages in the learning process. Teachers need to promote a responsible attitude toward these self-assessment activities. Becoming effective self-assessors is a crucial part of becoming autonomous learners. By instructing, motivating, providing positive reinforcement, and systematically supervising, the learning facilitator will help students develop a positive attitude toward their own progress.

For variation, students may be paired and peer-assessing may become part of the system. The teacher may decide to have the student self-assess some of the activities, have a peer assess other activities, and become directly involved in assessing the remainder of the activities.

When the activities have been assessed, the student should be directed to make corrections. This should be made clear to students right from the start. It is important to note the correct association between the question and the response to clarify understanding, aid retention, and be of use for study purposes.

Many of the activities include choices for the student. If the student is having difficulty, more practice may be warranted, and the student may need to be encouraged to do more of the choices.

Each section within a module includes additional types of activities called Extra Help and Enrichment. Students are expected to be involved in the decision as to which pathway best suits their needs. They may decide to do both.

Self-appraisal techniques can also be introduced at the individual conferences. Such questions as the following might be included:

- What steps are you taking to improve your understanding of this topic?
- What method of study do you use most?
- How do you organize your material to remember it?
- What steps do you follow when doing an assignment?
- What could you do to become an even better reader?
- Do you have trouble following directions?
- Did you enjoy this module?

A chart or checklist could be used for recording responses.

Informal Evaluation: Assignments

Informal evaluation, such as the assignments included in each module, are an invaluable aid to the teacher. They offer ongoing assessment information about the student's achievement and the behaviour and attitudes that affect that achievement.

Each module contains a separate booklet called the Assignment Booklet. This booklet assesses the knowledge or skills that the student has gained from the module. **The student's mark for the module may be based solely on the outcome of learning evident in the assignment booklet; however, you may decide to establish a value for other variables such as attitude or effort.** It is important that you establish at the beginning which outcomes will be evaluated, and that all students clearly understand what is expected.

Final Test

All LFMs include a formal final test which can be photocopied for each member of the class. The test, closely linked to the learning outcomes stated in the module booklets, gives the teacher precise information concerning what each student can or cannot do. Answers, explanations, and marking guides are also included. The value of the final test and each module is the decision of the classroom teacher. Following is a suggestion only.

Module 1 **6%**	**Module 2** **6%**	**Module 3** **9%**	**Module 4** **9%**
Module 5 **10%**	**Module 6** **9%**	**Module 7** **13%**	**Module 8** **8%**
	Final Test **30%**		

Introducing Students to the System

Your initiation to these learning materials began with a basic survey of what was included and how the components varied. This same process should be used with the class. After the materials have been explored, a discussion might include the advantages and the disadvantages of learning independently or in small groups. The roles of the students and teacher should be analysed. The necessary progress checks and rules need to be addressed. Your introduction should motivate students and build a responsible attitude toward learning autonomously.

Skill Level

It is important for students to understand that there are certain skills that they will need in order to deal successfully with the course materials. They are listed below:

- understanding and using instructional materials (table of contents, appendices, and glossary)
- applying basic algebraic operations and fundamental mathematical concepts
- employing problem-solving stategies
- recognizing and using special symbols
- using a graphing calculator

Other general skills are using reliable study methods, outlining, and learning to read at a flexible rate.

To decide the level and amount of instruction needed to accommodate the varied levels among students, you may wish to prepare and administer skill inventories or pretests. If most students need help with a particular skill, you may want to plan a total class instructional session. If only certain students lack a skill, you may want to set up a temporary skill group to help students who need it, or you may want to develop a skills file for this purpose.

Reading Level

These course materials are largely print based, but poorer readers need not be discouraged. It is important that you assure the students that these materials have been designed for easy reading. The authors have employed special strategies that lower and control the reading level. Some of them are

- the conscious selection of vocabulary and careful structuring of sentences to keep the materials at an independent reading level

- the integration of activities, examples, and illustrations to break text into appropriate-sized chunks

- the inclusion of many kinds of organizers (advance, graphic, intermediate, concept mapping, post organizers) to help give students a structure for incorporating new concepts

- the recognition that vocabulary and concepts are basic to understanding content materials and, thus, must be handled systematically (defined in context, marginal notes, footnotes, and often in a specialized glossary)

- the acknowledgement that background knowledge and experience play a vital role in comprehension

- the systematic inclusion of illustrations and videos to help poorer readers and visual learners, and audiocassettes and software as an alternative to print-based learning

- a variety of formats (paragraphs, lists, charts, etc.) to help poorer readers who do not absorb or retain main ideas easily in paragraph format

- the inclusion of media and activity choices to encourage an active rather than passive approach

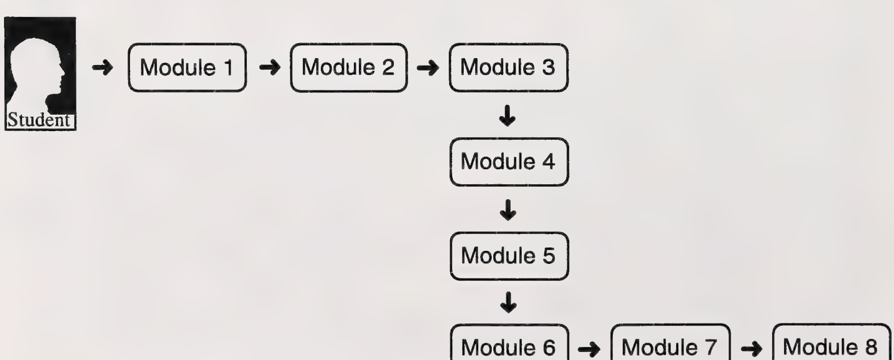
- instruction in a meaningful setting rather than in a contrived, workbook style

- using purposeful reading, viewing, and doing to produce better interpretation of the course materials

- the recognition that students need structured experiences when reading, viewing, or listening to instructional materials: developing pupil readiness, determining the purpose, providing guided instruction and feedback, rereading if necessary, and extending (This structure closely resembles the reading process.)

To help make the learning package more readable, you can begin your module preparation by reading (viewing, listening to) all the related materials that are going to be used. You need a solid background in order to assess and develop a background knowledge for students. The students' experiential bases may be assessed through brainstorming sessions concerning the topic, or by using visuals and guided questions to predict what the topic might be about.

It is recommended that you start with Module 1 because this module includes basic introductory information, and it is also recommended that you do the modules sequentially—ending with Module 8 because this module acts as a summary or culmination.

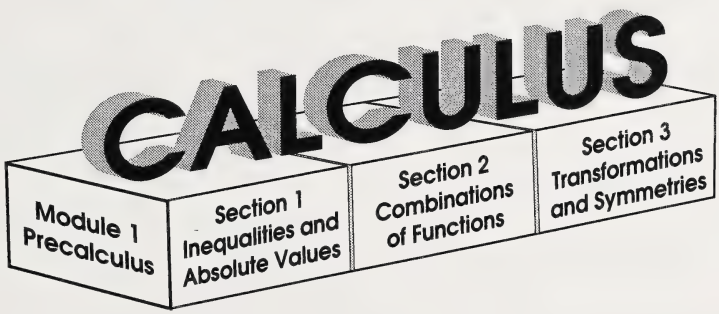


Module 1: Precalculus

Overview

Calculus builds on skills and concepts studied in previous mathematics courses. The algebra of polynomials, factoring, operations with rational expressions, exponents and radicals, and equation solving are but a few of the foundation skills required for Mathematics 31. Module 1 presents additional skills basic to calculus.

Section 1 introduces interval notation, reviews absolute value, and develops techniques for solving a variety of inequalities. Section 2 introduces the algebra of functions: how functions can be combined through composition and using the arithmetic operations of addition, subtraction, multiplication, and division. Section 3 builds on the previous two sections dealing specifically with the analysis and transformation of the graphs of functions.



Evaluation

The evaluation of this module will be based on four assignments:

Section 1 Assignment	**38 marks**
Section 2 Assignment	**26 marks**
Section 3 Assignment	**26 marks**
Final Module Assignment	**10 marks**
TOTAL	**100 marks**

Section 1: Inequalities and Absolute Values

Key Concepts

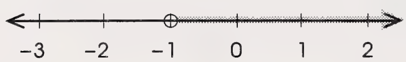
- interval notation
- absolute value
- inequalities

The basic goals of this section are to ensure that students

- write and interpret the solutions to inequalities using set-builder notation, a graph, or interval notation
- solve inequalities of the type $|P(x)| \geq a$ or $|P(x)| \leq a$, $\frac{P(x)}{Q(x)} \geq a$ or $\frac{|P(x)|}{|Q(x)|} \geq a$, and $ax^2 + bx + c \geq d$

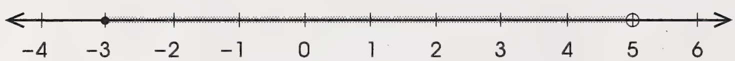
Section 1: Assignment Answer Key (38 marks)

1. $(-1, \infty)$



(2 marks)

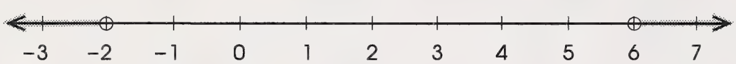
2. $\{x \mid -3 \le x < 5\}$



(2 marks)

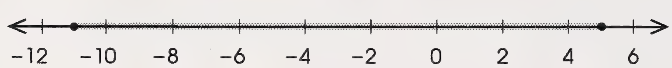
3.
$$\begin{aligned} 4-2x &> 8 & \text{or} \quad 4-2x &< -8 \\ -2x &> 4 & -2x &< -12 \\ x &< -2 & x &> 6 \end{aligned}$$

The solution set is $(-\infty, -2) \cup (6, \infty)$.



(5 marks)

4.
$$-8 \le x+3 \le 8$$
$$-11 \le x \le 5$$
$$[-11, 5]$$



(5 marks)

5.
$$\frac{x}{x-2} - 2 > 0$$
$$\frac{x-2(x-2)}{x-2} > 0$$
$$\frac{-x+4}{x-2} > 0$$

Case 1: $-x+4>0$ and $x-2>0$
$x<4$ $\quad x>2$

Thus, $(2, 4)$ is part of the solution set.

Case 2: $-x+4<0$ and $x-2<0$
$x>4$ $\quad x<2$

Case 2 has no solution.

Therefore, the solution set is $(2, 4)$. **(9 marks)**

6. $|x| < 2|x+2|$

Case 1: Assume $x \geq 0$ and $x+2 > 0$. Hence, $x \geq 0$.

$$\begin{aligned} \therefore\ x &< 2(x+2) \\ x &< 2x+4 \\ x &> -4 \end{aligned}$$

Reconciling this solution with your assumption, you see that $x \geq 0$.

Case 2: Assume $x < 0$ and $x+2 < 0$. Hence, $x < -2$.

$$\begin{aligned} \therefore\ -x &< 2(-x-2) \\ -x &< -2x-4 \\ x &< -4 \end{aligned}$$

Reconciling this solution with your assumption, you see that $x < -4$.

Case 3: Assume $x < 0$ and $x+2 > 0$. Hence, $-2 < x < 0$.

$$\begin{aligned} \therefore\ -x &< 2(x+2) \\ -x &< 2x+4 \\ x &> -\frac{4}{3} \end{aligned}$$

Reconciling this solution with your assumption, you see that $-\frac{4}{3} < x < 0$.

Case 4: Assume $x > 0$ and $x+2 < 0$. This is impossible.

The solution set is $(-\infty, -4) \cup \left(-\frac{4}{3}, 0\right) \cup [0, \infty)$ or $(-\infty, -4) \cup \left(-\frac{4}{3}, \infty\right)$. **(11 marks; accept equivalent answers)**

7. $x^2 - 3x - 4 > 0$
$(x-4)(x+1) > 0$

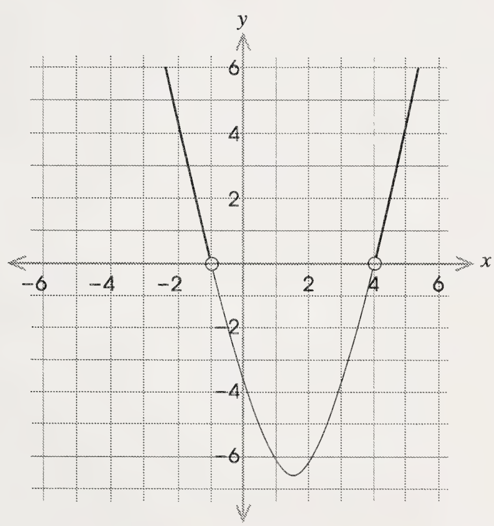
Case 1: $x-4 > 0$ and $x+1 > 0$
$x > 4$ $\qquad$ $x > -1$

Therefore, $x > 4$.

Case 2: $x-4 < 0$ and $x+1 < 0$
$x < 4$ $\qquad$ $x < -1$

Therefore, $x < -1$.

The solution set is $(4, \infty) \cup (-\infty, -1)$. **(4 marks)**



Section 2: Combinations of Functions

Key Concepts

- the sum, difference, product, and quotient of two functions
- composition of two functions

The basic goals of this section are to ensure that students

- combine functions algebraically, using the arithmetic operations of addition, subtraction, multiplication, and division
- express complex functions as combinations of simpler functions
- determine the composition of two functions
- analyse and draw the graphs of combinations of functions
- model everyday situations using combinations of functions

Section 2: Assignment Answer Key (26 marks)

1. $(f+g)(x)=\sqrt{x+1}-x$

The domain of $f+g$ is $\{x|x\geq -1\}$.

$(f-g)(x)=\sqrt{x+1}+x$

The domain of $f-g$ is $\{x|x\geq -1\}$.

$(fg)(x)=-x\sqrt{x+1}$

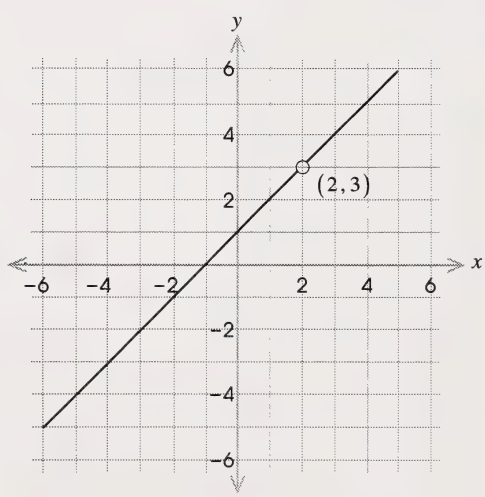
The domain of fg is $\{x|x\geq -1\}$.

$\left(\frac{f}{g}\right)(x)=\frac{\sqrt{x+1}}{-x}$

The domain of $\frac{f}{g}$ is $\{x|x\geq -1 \text{ and } x\neq 0\}$.

(4 marks)

2. $$\left(\frac{f}{g}\right)(x)=\frac{x^2-x-2}{x-2}$$
$$=\frac{(x-2)(x+1)}{x-2}$$
$$=x+1,\text{ where } x\neq 2 \quad \textbf{(3 marks)}$$



3. **a.** $g(4)=4-3$
$=1$

$f(g(4))=f(1)$
$=1^2-1-2$
$=-2$ **(1 mark)**

b. $g(0)=0-3$
$=-3$

$h(g(0))=h(-3)$
$=2(-3)$
$=-6$

$f(h(g(0)))=f(-6)$
$=(-6)^2-(-6)-2$
$=40$ **(2 marks)**

c. $f(h(x))=f(2x)$
$=(2x)^2-2x-2$
$=4x^2-2x-2$

$g(f(h(x)))=g(4x^2-2x-2)$
$=4x^2-2x-2-3$
$=4x^2-2x-5$ **(2 marks)**

4. Let $f(x)=\frac{3-x}{5-x}$ and $g(x)=x^3$.

Therefore, $F(x)=f(g(x))$. **(2 marks)**

5. $A(x)=4\pi x^2 \qquad (x=2t)$
$=4\pi(2t)^2$
$=16\pi t^2$

The balloon's surface area as a function of time t is $16\pi t^2$. **(2 marks)**

6. $g(x)=\sqrt{2-x}$ is defined for $x\leq 2$.

$f(g(x))=\sqrt{1-\sqrt{2-x}}$ would be defined if $\sqrt{2-x}\leq 1$
$2-x\leq 1$
$x\geq 1$

Therefore, the domain is $[1,2]$. **(4 marks)**

7. $f(x)=g(h(x))$
$x^2-x+2=h(x)-2$
$h(x)=x^2-x+4$ **(2 marks)**

8. **Method 1**

$$g(g(x)) = g(x) - 2$$
$$= (x-2) - 2$$
$$= x - 4$$

$$f(f(x)) = 3(f(x)) - 2$$
$$= 3(3x-2) - 2$$
$$= 9x - 8$$

$$\therefore x - 4 = 9x - 8$$
$$8x = 4$$
$$x = \frac{4}{8}$$
$$x = \frac{1}{2} \text{ or } 0.5$$ **(4 marks)**

Method 2

$$g(g(x)) = f(f(x))$$
$$g(x) - 2 = 3(f(x)) - 2$$
$$(x-2) - 2 = 3(3x-2) - 2$$
$$x - 4 = 9x - 8$$
$$8x = 4$$
$$x = \frac{8}{4}$$
$$= \frac{1}{2} \text{ or } 0.5$$

Section 3: Transformations and Symmetries

Key Concepts

- vertical and horizontal translations
- reflection in the x-axis, the y-axis, and the line $y = x$
- inverse functions
- vertical scaling: dilations and compressions
- symmetry across the x-axis, the y-axis, and the origin

The basic goals of this section are to ensure that students

- sketch and describe any translation, reflection or dilation on a function or its inverse

  These functions include the linear, quadratic and cubic functions; the absolute value function; the reciprocal function; and the exponential function.

- describe the effects of the parameters on the general function $y = af(x \pm c) \pm d$

- alter the equation of a function or relation to reflect it across the axes or to reflect it across the line $y = x$

- graph of the inverse of a function

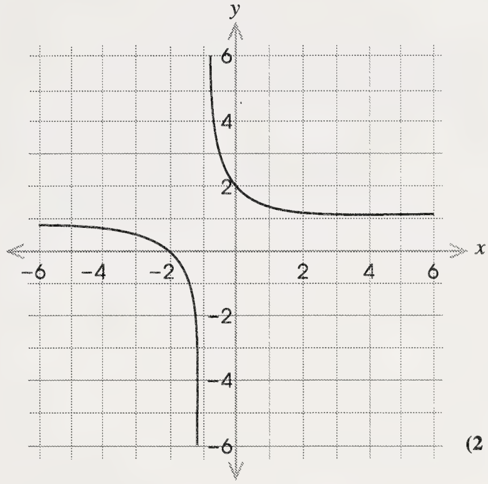
- recognize the conditions under which symmetry occurs; symmetry in the x-axis, the y-axis, or the origin

Section 3: Assignment Answer Key (26 marks)

1. The equation is $y = \sqrt{x+4} - 2$. **(2 marks)**

2. The graphs of $y = f(x-5) + 4$ is the graph of $y = f(x)$ translated 5 units to the right and 4 units up. **(2 marks)**

3. The new equation is $y=\frac{1}{x+1}+1$. **(1 mark)**



(2 marks)

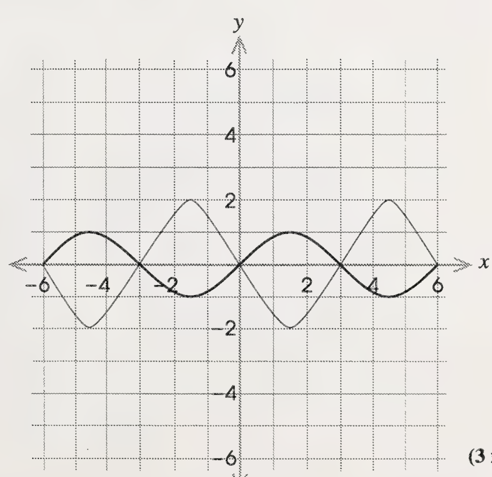
4. **a.** *m* is odd and *n* is even. **(1 mark)** **b.** *m* is even and *n* is odd. **(1 mark)**
c. *m* and *n* are even. **(1 mark)**

5. **a.** $y=-x^3-2x^2$ **(1 mark)** **b.** $y=-x^3+2x^2$ **(1 mark)**
c. $x=y^3+2y^2$ **(1 mark)**

6. The reflection $x=3y-5$ is a linear function because for each value of x, y is uniquely determined.

$$x=3y-5$$
$$y=\frac{x+5}{3}$$
$$f^{-1}(x)=\frac{x+5}{3}$$ **(3 marks)**

7. The graph must show the same x-intercepts, be twice as tall, and be flipped about the x-axis.



(3 marks)

8. The graph $y=x^2$ has been transformed as follows:

- a horizontal translation of 1 unit to the left
- a vertical translation of 3 units upward
- stretched by a factor of 2
- flipped over

Therefore, the equation of the graph is $y=-2(x+1)^2+3$. **(5 marks)**

9. The reflection in the x-axis is $y=-x^2+1$.

Thus, $x^2-1=-x^2+1$

$$2x^2=2$$

$$x=\pm 1$$

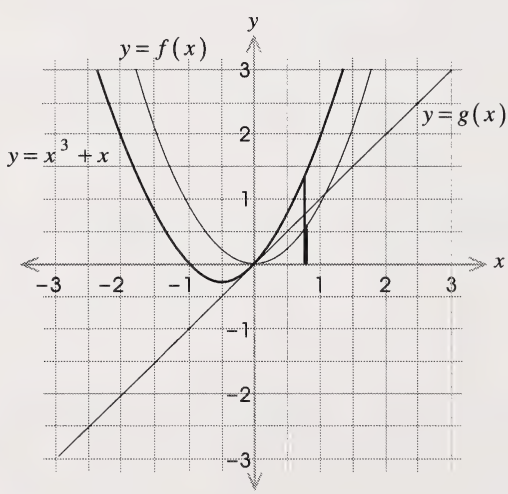
The common points are $(\pm 1, 0)$. **(2 marks)**

Final Module Assignment (10 marks)

1. No, an endpoint of the interval is included in the set. **(2 marks)**

2. $y=x^2+x$ can be viewed as $y=f(x)+g(x)$.

The y-coordinates of the points on the graph of $y=x^2+x$ are the sums of the y-coordinates of points of the graphs of $f(x)=x^2$ and $g(x)=x$ as shown. **(4 marks)**

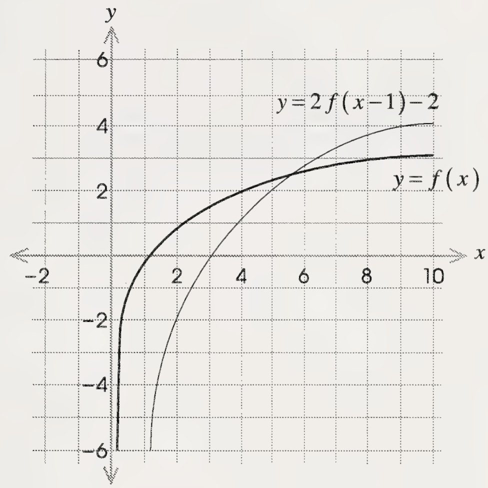


Give one mark for accuracy and one mark for some indication that $y=x^2+x$ is a sum.

3. The transformations are as follows:

- a horizontal translation of 1 unit to the right
- a vertical translation of 2 units downward
- stretched by a factor of 2

(4 marks)



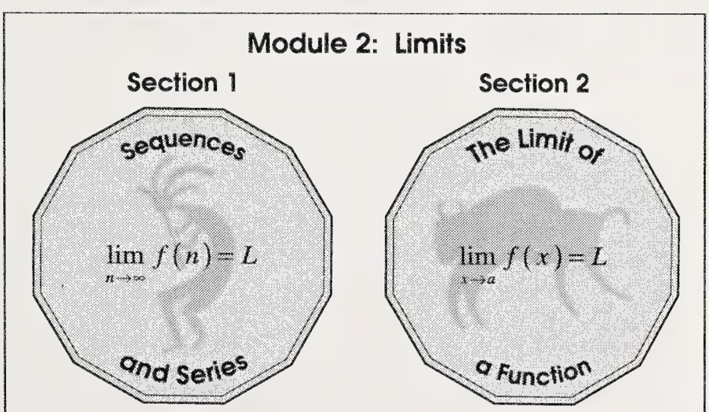
Module 2: Limits

Overview

In this module, the student is introduced to the topic of limits from two complementary points of view.

Section 1 uses sequences and series to develop the concept of limit, and to provide the student with tools to find limits numerically and geometrically. In particular, the student is expected to determine the limits of convergent infinite sequences. As well, the student explores infinite geometric series, applies techniques for evaluating those series, and uses series to model everyday situations.

Section 2 uses algebraic functions and their graphs to extend the student's understanding of limits. This section defines limits, introduces techniques for determining left- and right-hand limits, discusses continuous and discontinuous functions, uses limit theorems to find limits of algebraic functions, and inspects curves at their extremes.



Evaluation

The evaluation of this module will be based on three assignments:

Section 1 Assignment	**32 marks**
Section 2 Assignment	**46 marks**
Final Module Assignment	**22 marks**
TOTAL	**100 marks**

Mandatory Video

Section 2: Activity 5

An Introduction to Calculus and Vectors, *Catch 31* series, ACCESS Network, available from Learning Resources Distributing Centre.

Section 1: Sequences and Series

Key Concepts

- infinite sequences
- convergent and divergent sequences
- limit of a sequence
- infinite series
- infinite geometric series
- test for convergence of an infinite geometric series

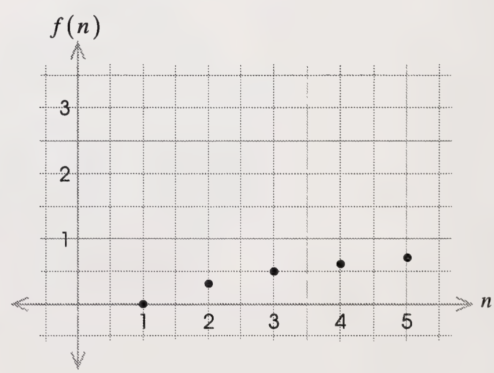
The basic goals of this section are to ensure that students

- define an infinite sequence
- test for convergence and divergence
- determine limits of infinite sequences from their graphs or algebraically
- define an infinite series
- apply the test for convergence of an infinite geometric series
- determine the sum of a convergent infinite geometric series
- apply their knowledge of sequences, series, and limits in a variety of contexts

Section 1: Assignment Answer Key (32 marks)

1. **a.** $f(1)=\frac{1-1}{1+1}=0$ $\qquad f(2)=\frac{2-1}{2+1}=\frac{1}{3}$ $\qquad f(3)=\frac{3-1}{3+1}=\frac{2}{4}=\frac{1}{2}$

$f(4)=\frac{4-1}{4+1}=\frac{3}{5}$ $\qquad f(5)=\frac{5-1}{5+1}=\frac{4}{6}=\frac{2}{3}$



This is a convergent sequence with a limit of 1. **(6 marks)**

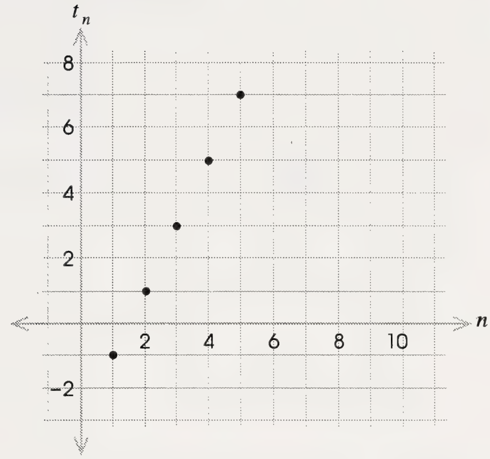
b. $t_1 = 2(1) - 3$
$= 2 - 3$
$= -1$

$t_2 = 2(2) - 3$
$= 4 - 3$
$= 1$

$t_3 = 2(3) - 3$
$= 6 - 3$
$= 3$

$t_4 = 2(4) - 3$
$= 8 - 3$
$= 5$

$t_5 = 2(5) - 3$
$= 10 - 3$
$= 7$



This is a divergent sequence since $t_n \to \infty$ as $n \to \infty$. **(6 marks)**

2. 50, $50(0.8)$, $50(0.8)^2$, $50(0.8)^3$, $50(0.8)^4$ or 50, 40, 32, 25.6, 20.48

$t_n = 50(0.8)^{n-1}$, where $n \in N$ **(4 marks)**

3. **a.**
$$\lim_{n\to\infty} \frac{2n^2 - 3n + 1}{5 - 3n^2} = \lim_{n\to\infty} \frac{n^2\left(2 - \frac{3}{n} + \frac{1}{n^2}\right)}{n^2\left(\frac{5}{n^2} - 3\right)}$$
$$= \frac{2 - 0 + 0}{0 - 3}$$
$$= -\frac{2}{3}$$ **(3 marks)**

b.
$$\lim_{n\to\infty} \frac{6n^3 - 5}{n - 4} = \lim_{n\to\infty} \frac{n^3\left(6 - \frac{5}{n^3}\right)}{n^3\left(\frac{1}{n^2} - \frac{4}{n^3}\right)}$$
$$= \frac{6 - 0}{0 - 0}$$
= undefined **(3 marks)**

4. **a.**
$$S = \frac{a}{1 - r}$$
$$= \frac{-\frac{1}{2}}{1 - \left(-\frac{1}{2}\right)}$$
$$= \frac{-\frac{1}{2}}{\frac{3}{2}}$$
$$= -\frac{1}{3}$$ **(3 marks)**

b.
$$S = \frac{a}{1 - r}$$
$$= \frac{1}{1 - (1 + x)}$$
$$= \frac{1}{1 - 1 - x}$$
$$= -\frac{1}{x}$$ **(3 marks)**

5. $0.324\,545\ldots = 0.32 + \left(0.0045 + 0.000\,045 + 0.000\,0045\right)$
$a = 0.0045$ and $r = 0.001$

$$\begin{aligned} S &= 0.32 + \frac{a}{1-r} \\ &= \frac{32}{100} + \frac{0.0045}{1-0.01} \\ &= \frac{32}{100} + \frac{0.0045}{0.99} \\ &= \frac{32}{100} + \frac{45}{9900} \\ &= \frac{3213}{9900} \\ &= \frac{357}{1100} \quad \textbf{(4 marks)} \end{aligned}$$

Section 2: The Limit of a Function

Key Concepts

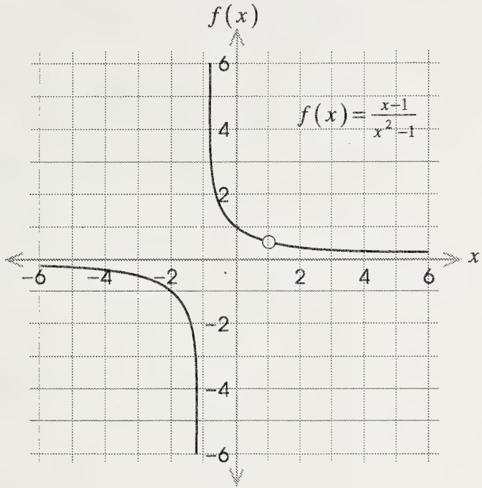
- limits of algebraic functions
- limit theorems
- left-hand and right-hand limits
- limits of $f(x)$ as $x \to \pm\infty$
- continuity and discontinuity
- horizontal asymptotes

The basic goals of this section are to ensure that students

- explain the concept of a limit
- give examples of functions with limits, with left-hand or right-hand limits, or with no limits
- determine the limits of algebraic functions as the independent variable approaches finite or infinite values for both continuous and discontinuous functions
- sketch continous and discontinuous functions using limits
- give examples of bounded and unbounded functions, and of bounded functions with no limits
- explain, illustrate, and use the limit theorems for sums, differences, multiples, products, quotients, powers, and roots

Section 2: Assignment Answer Key (46 marks)

1.



$$\lim_{x \to 1} f(x) = \frac{1}{2}$$

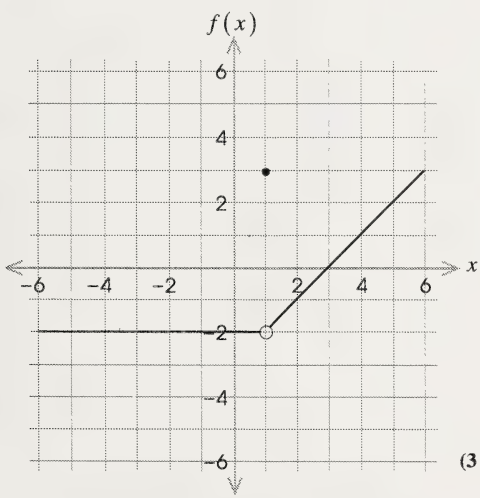
The function is discontinuous at the following:

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$ **(5 marks)**

2. a.



(3 marks)

b. $\lim_{x \to 1^-} f(x) = -2$ $\lim_{x \to 1^+} f(x) = x - 3 = -2$ $\lim_{x \to 1} f(x) = -2$ **(3 marks)**

c. The function is discontinuous at $x = 1$ since $f(1) \neq -2$. **(2 marks)**

3. a. $\lim_{x \to 4^-} f(x) = 4$ **(1 mark)** **b.** $\lim_{x \to 4^+} f(x) = 6$ **(1 mark)** **c.** $\lim_{x \to 4} f(x) = \text{undefined}$ **(1 mark)**

4. **a.** $\lim\limits_{x\to 3}(5x-3)=5(3)-3$

$=12$ **(2 marks)**

b. $\lim\limits_{x\to 2} 3=3$ **(1 mark)**

c. $\lim\limits_{x\to 2}\dfrac{x-2}{x^3-8}=\lim\limits_{x\to 2}\dfrac{\cancel{(x-2)}}{\cancel{(x-2)}\left(x^2+2x+4\right)}$

$=\dfrac{1}{2^2+2(2)+4}$

$=\dfrac{1}{12}$ **(3 marks)**

d. $\lim\limits_{x\to 1}\dfrac{x-1}{1-x^2}=\lim\limits_{x\to 1}\dfrac{x-1}{(1-x)(1+x)}$

$=\lim\limits_{x\to 1}\dfrac{-\cancel{(1-x)}}{\cancel{(1-x)}(1+x)}$

$=\dfrac{-1}{1+1}$

$=-\dfrac{1}{2}$ **(3 marks)**

e. $\lim\limits_{x\to 3}\dfrac{\frac{1}{x}-\frac{1}{3}}{x-3}=\lim\limits_{x\to 3}\dfrac{\frac{(3-x)}{3x}}{x-3}$

$=\lim\limits_{x\to 3}\dfrac{\frac{-\cancel{(x-3)}}{3x}}{\cancel{(x-3)}}$

$=\lim\limits_{x\to 3}\dfrac{-1}{3x}$

$=-\dfrac{1}{9}$ **(3 marks)**

f. $\lim\limits_{x\to 0}\dfrac{\sqrt{x+4}-2}{x}=\lim\limits_{x\to 0}\dfrac{\left(\sqrt{x+4}-2\right)\left(\sqrt{x+4}+2\right)}{x\left(\sqrt{x+4}+2\right)}$

$=\lim\limits_{x\to 0}\dfrac{\left(\sqrt{x+4}\right)^2-4}{x\left(\sqrt{x+4}+2\right)}$

$=\lim\limits_{x\to 0}\dfrac{(x+4-4)}{x\left(\sqrt{x+4}+2\right)}$

$=\lim\limits_{x\to 0}\dfrac{\cancel{x}}{\cancel{x}\left(\sqrt{x+4}+2\right)}$

$=\lim\limits_{x\to 0}\dfrac{1}{\sqrt{x+4}+2}$

$=\dfrac{1}{\sqrt{4}+2}$

$=\dfrac{1}{4}$ **(4 marks)**

5.

LS	RS
$\lim\limits_{x\to 2}\left(5-3x^2\right)$	$\lim\limits_{x\to 2} 5-3\left(\lim\limits_{x\to 2} x\right)^2$
$=5-3(2)^2$	$=5-3(2)^2$
$=5-12$	$=5-12$
$=-7$	$=-7$

LS = RS **(3 marks)**

6. **a.** $\lim\limits_{x\to\infty}\dfrac{(x-1)}{(1-x)}=\lim\limits_{x\to\infty}\dfrac{-1\cancel{(1-x)}}{\cancel{(1-x)}}$

$=-1$ **(2 marks)**

b. $\lim\limits_{x\to-\infty}(x-2)(x-3)=\infty$

Both factors are negative and increasing in absolute value. **(2 marks)**

c. $\lim\limits_{x\to-\infty} 4^x=0$

The value x is a negative and increasing. **(2 marks)**

d. $\lim\limits_{x\to 3}\dfrac{x^2-2x-3}{2x^2-7x+3}=\lim\limits_{x\to 3}\dfrac{\cancel{(x-3)}(x+1)}{(2x-1)\cancel{(x-3)}}$

$=\lim\limits_{x\to 3}\dfrac{x+1}{2x-1}$

$=\dfrac{3+1}{2(3)-1}$

$=\dfrac{4}{5}$ **(3 marks)**

e. $\lim\limits_{x\to\infty}\dfrac{2x^2}{x^2-2}=\lim\limits_{x\to\infty}\dfrac{2x^2}{x^2\left(1-\frac{2}{x^2}\right)}$

$=\dfrac{2}{1-0}$

$=2$ **(2 marks)**

Final Module Assignment (22 marks)

1. Answers will vary.

A representative answer is $f(n)=(-1)^n$, where $n\in N$. Since the terms alternate between -1 and $+1$, there is no limit. **(2 marks)**

2. For the series to converge, $|2+x|<1$.

$$\therefore \quad -1<2+x<1$$
$$-3<x<-1 \quad \textbf{(2 marks)}$$

3. $S=\dfrac{a}{1-r}$

$=\dfrac{100}{1-0.9}$

$=\dfrac{100}{0.1}$

$=1000$

A total of \$1000 is lent out.

Interest $=\dfrac{10}{100}\times 1000$

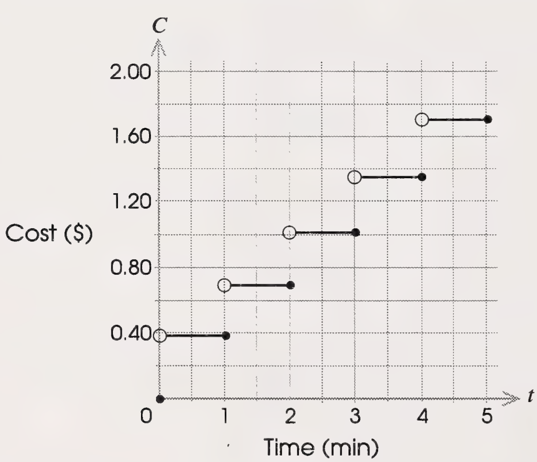
$=100$

The interest paid is \$100. **(4 marks)**

4. **a.** $\lim\limits_{x\to 2^-} f(x)=2$ **(1 mark)** **b.** $\lim\limits_{x\to 2^+} f(x)=2$ **(1 mark)** **c.** $\lim\limits_{x\to 2} f(x)=2$ **(1 mark)**

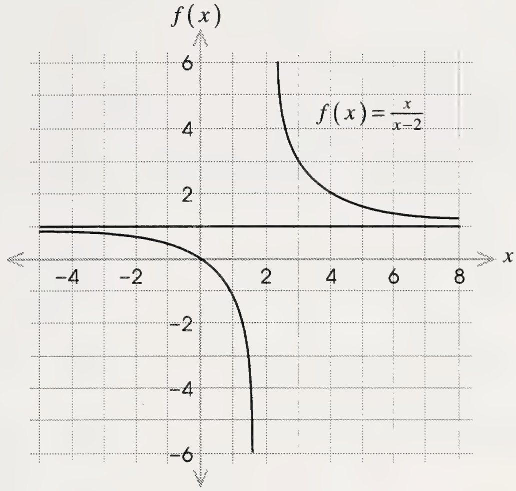
d. The function is discontinuous at 1 and 2. **(2 marks)**

5.



This is a discontinuous function. The discontinuities occur at $t=0$ min, $t=1$ min, $t=2$ min, $t=3$ min, and so on. **(5 marks)**

6. Since $\lim\limits_{x \to \pm\infty} \frac{x}{x-2} = 1$, $y = 1$ is a horizontal asymptote. **(4 marks)**

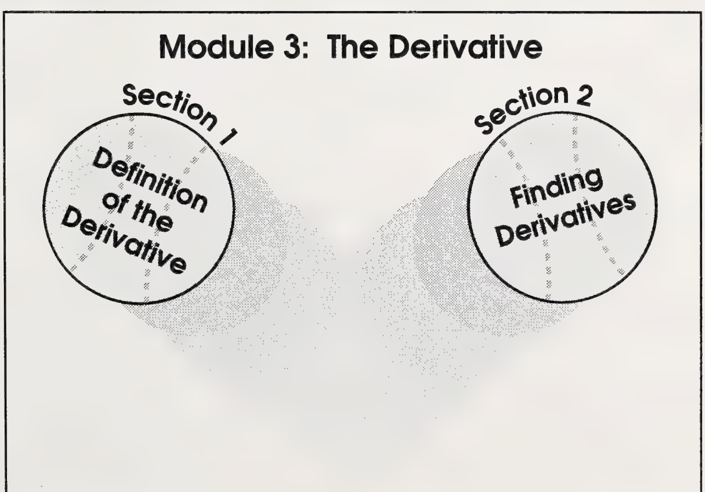


Module 3: The Derivative

Overview

This module introduces the student to differential calculus, in particular, the derivative and the techniques for determining the derivative of algebraic functions.

Section 1 deals with the definition of the derivative and its symbols. The student explores the relationships between the derivative and secants, tangents, and normals drawn to the graphs of functions. In Section 2, the student explores the methods of finding derivatives of algebraic functions: specifically, the derivative of a constant, the derivative of the product of a constant and a function, the power rule, the sum and difference rules, the product and quotient rules, and the chain rule. As well, the student is introduced to implicit differentiation and to higher-order derivatives. These methods are essential to the student's further study and appreciation of calculus's application to problems in the sciences and in everyday situations.



Evaluation

The evaluation of this module will be based on three assignments:

Section 1 Assignment	**25 marks**
Section 2 Assignment	**55 marks**
Final Module Assignment	**20 marks**
TOTAL	**100 marks**

Materials

Section 1: Activity 2

The materials needed are as follows:

- 15 cm by 15 cm construction paper
- scissors
- tape

Mandatory Videos

Section 2: Activity 2

The Derivative/The Power Rule, *Catch 31* series, ACCESS Network, available from Learning Resources Distributing Centre.

Section 2: Activity 4

The Sum Rule and the Chain Rule for Derivatives, *Catch 31* series, ACCESS Network, available from Learning Resources Distributing Centre.

Section 2: Activity 5

The Product Rule and the Quotient Rule, *Catch 31* series, ACCESS Network, available from Learning Resources Distributing Centre.

Section 2: Activity 6

The Sum Rule and the Chain Rule for Derivatives, *Catch 31* series, ACCESS Network, available from Learning Resources Distributing Centre.

Section 2: Activity 7

The Product Rule and the Quotient Rule, *Catch 31* series, ACCESS Network, available from Learning Resources Distributing Centre.

Section 2: Activity 8

Derivatives for Relations or Functions Defined Implicitly, *Catch 31* series, ACCESS Network, available from Learning Resources Distributing Centre.

Section 1: Definition of the Derivative

Key Concepts

- secant
- tangent
- normal
- derivative

The basic goals of this section are to ensure that students

- find the slope and equation of a secant line
- show that the slope of a tangent line is a limit
- explain how the derivative can be approximated by a sequence of secant lines
- estimate the numerical value of the derivative at a point using a sequence of secant lines
- explain how the derivative of a function is connected to the slope of the tangent line
- find the slopes and equations of tangent lines at given points on a curve, using the definition of a derivative
- determine the slopes and equations of normal lines

Section 1: Assignment Answer Key (25 marks)

1. When $x=1$, $y=1^2-1$
$=0$

When $x=2$, $y=2^2-2$
$=4-2$
$=2$

$$\begin{aligned}\text{slope} &= \frac{y_2-y_1}{x_2-x_1}\\ &= \frac{2-0}{2-1}\\ &= 2\end{aligned}$$

The slope of the secant line is 2. **(4 marks)**.

2. When $x=-1$, $y=(-1)^2+2(-1)-3$
$=-4$

When $x=2$, $y=2^2+2(2)-3$
$=5$

$$\begin{aligned}\text{slope} &= \frac{y_2-y_1}{x_2-x_1}\\ &= \frac{5-(-4)}{2-(-1)}\\ &= 3\end{aligned}$$

Use $y-y_1=m(x-x_1)$ to find the equation of the secant.

$$\begin{aligned}y-y_1 &= m(x-x_1)\\ y-5 &= 3(x-2)\\ y-5 &= 3x-6\\ 3x-y-1 &= 0\end{aligned}$$

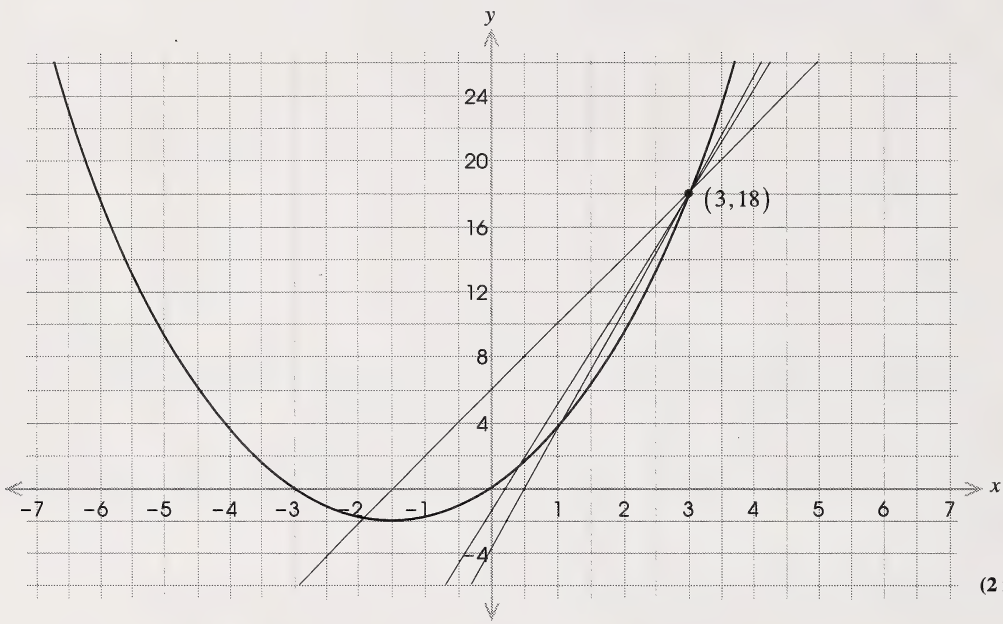
The equation of the secant is $3x-y-1=0$. **(6 marks)**

3. a. When $x=3$, $y=x^2+3x$
$=3^2+3(3)$
$=9+9$
$=18$

When $x=3+h$, $y=(3+h)^2+3(3+h)$
$=9+6h+h^2+9+3h$
$=18+9h+h^2$

$$\begin{aligned} m_{\text{sec}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\left(18 + 9h + h^2\right) - 18}{(3+h) - 3} \\ &= \frac{9h + h^2}{h} \\ &= 9 + h \end{aligned}$$ **(5 marks)**

b.



(2 marks)

The sequence illustrated is only one possible sequence. Here, h is shown as negative; accept sequences shown for positive values of h as well.

c. Because the slope of the secant line is given by $9 + h$, as h approaches 0, the slopes of the sequence of secant lines approaches 9. The slope of the tangent at $(3, 18)$ is 9. **(2 marks)**

4. When $x = 1$, $y = 1^3 - 1$
$= 0$

Find the equation of the tangent.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= 2(x - 1) \\ y &= 2x - 2 \end{aligned}$$

Now find the equation of the normal. The slope of the normal is $-\frac{1}{2}$.

$$\begin{aligned} y - y_1 &= m\left(x - x_1\right) \\ y - 0 &= -\frac{1}{2}(x-1) \\ y &= -\frac{1}{2}x + \frac{1}{2} \end{aligned}$$

The equation of the tangent is $y = 2x - 2$, and the equation of the normal is $y = -\frac{1}{2}x + \frac{1}{2}$. **(6 marks)**

Section 2: Finding Derivatives

Key Concepts

- first principles
- differentiable and non-differentiable functions
- power rule
- the derivative of $c \cdot f(x)$
- sum and difference rule
- product rule
- quotient rule
- chain rule
- implicit differentiation
- higher-order derivatives

The basic goals of this section are to ensure that students

- find the derivative from first principles
- interpret a derivative as a slope, rate, or function
- explain circumstances when the derivative does or does not exist
- identify and use the notations $f'(x)$, y', and $\frac{dy}{dx}$ as alternate notations for the first derivative of a function
- differentiate functions of the type $y = x^n$ where n is rational
- demonstrate the need for and apply the sum and difference rules for derivatives, and the rule for finding the derivative of the product of a constant and a function
- demonstrate that the chain, power, product, and quotient rules are aids to differentiate complicated functions, and understand the derivation of those rules; explain the product and quotient rules using practical examples
- apply the power, product, quotient, and chain rules in combination
- derive the quotient rule from the product rule
- show that equivalent forms of a rational function can be found by using the product rule and quotient rule
- determine the derivative of a function expressed as a product of more than two factors
- write final answers in factored form
- find the slopes and equations of tangents and normals at a given point on a curve

- identify implicit differentiation as a tool for differentiating functions where one variable is difficult or impossible to isolate
- use the technique of implicit differentiation
- show the derivative for a relation found by both implicit and explicit differentiation to be the same
- find equations of tangents to conics
- illustrate and determine higher-order derivatives of algebraic functions
- describe the second derivative geometrically
- determine the second derivative of an implicitly defined function

Section 2: Assignment Answer Key (55 marks)

1. $f(x) = 2x^2 - 3$

$$\begin{aligned} f(x+h) &= 2(x+h)^2 - 3 \\ &= 2\left(x^2 + 2hx + h^2\right) - 3 \\ &= 2x^2 + 4hx + 2h^2 - 3 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad (\text{where } \Delta x = h) \\ &= \lim_{h \to 0} \frac{\left(2x^2 + 4hx + 2h^2 - 3\right) - \left(2x^2 - 3\right)}{h} \\ &= \lim_{h \to 0} \frac{4hx + 2h^2}{h} \\ &= \lim_{h \to 0} \frac{h(4x + 2h)}{h} \\ &= 4x + 0 \\ &= 4x \end{aligned}$$

(6 marks)

2. $y = 3x - \dfrac{5}{x}$

$$f(x) = 3x - \frac{5}{x}$$

$$f(x+h) = 3(x+h) - \frac{5}{x+h}$$

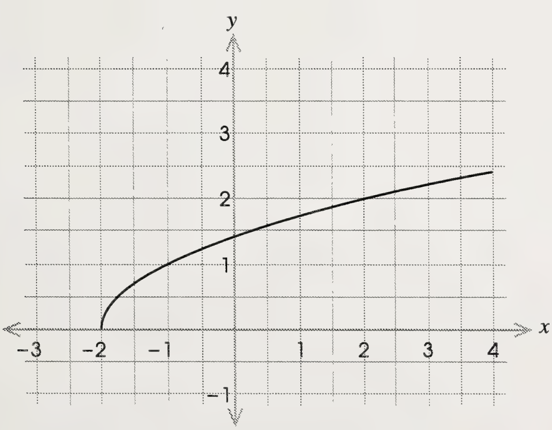
$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \; h \neq 0 \\
&= \lim_{h \to 0} \frac{3x + 3h - \dfrac{5}{x+h} - 3x + \dfrac{5}{x}}{h} \\
&= \lim_{h \to 0} \frac{3hx(x+h) - 5x + 5x + 5h}{x(x+h)h} \\
&= \lim_{h \to 0} \frac{3x\cancel{h}(x+h) + 5\cancel{h}}{x\cancel{h}(x+h)} \\
&= \lim_{h \to 0} \frac{3x(x+h) + 5}{x(x+h)}
\end{aligned}$$

$$\frac{dy}{dx} = \frac{3x^2 + 5}{x^2} \text{ or } 3 + \frac{5}{x^2}$$

Find the slope of the tangent at $x = 1$.

$$\begin{aligned}
f'(1) &= 3 + \frac{5}{1^2} \\
&= 3 + 5 \\
&= 8 \quad \textbf{(8 marks)}
\end{aligned}$$

3.



The derivative does not exist at $(-2, 0)$ because $\lim\limits_{h \to 0^-} \dfrac{f(-2+h) - f(-2)}{h}$ does not exist. **(3 marks)**

4. $y = 3x^4 - x^{-1}$

$\frac{dy}{dx} = 3(4x^3) - (-1x^{-2})$

$= 12x^3 + x^{-2}$

$f'(1) = 12(1)^3 + 1^{-2}$

$= 12 + 1$

$= 13$

Therefore, the slope of the normal is $-\frac{1}{13}$.

When $x = 1$, $y = 3(1)^4 - \frac{1}{1}$

$= 2$

The normal passes through $(1, 2)$.

$$y - y_1 = m(x - x_1)$$
$$y - 2 = -\frac{1}{13}(x - 1)$$
$$13y - 26 = -x + 1$$
$$x + 13y - 27 = 0$$

The equation of the normal is $x + 13y - 27 = 0$. **(8 marks)**

5. a. $y = (x-2)^3(2x-1)^{\frac{1}{2}}$

$\frac{dy}{dx} = (x-2)^3 \frac{d}{dx}(2x-1)^{\frac{1}{2}} + (2x-1)^{\frac{1}{2}} \frac{d}{dx}(x-2)^3$

$= (x-2)^3\left(\frac{1}{2}\right)(2x-1)^{-\frac{1}{2}}(2) + (2x-1)^{\frac{1}{2}}(3)(x-2)^2$

$= (x-2)^3(2x-1)^{-\frac{1}{2}} + (2x-1)^{\frac{1}{2}}(3)(x-2)^2$

$= (x-2)^2(2x-1)^{-\frac{1}{2}}[(x-2) + 3(2x-1)]$

$= (x-2)^2(2x-1)^{-\frac{1}{2}}[x - 2 + 6x - 3]$

$= (x-2)^2(2x-1)^{-\frac{1}{2}}(7x-5)$ **(6 marks)**

b. $y = \dfrac{(x-1)}{(2x+1)^{\frac{1}{2}}}$

$$\frac{dy}{dx} = \frac{(2x+1)^{\frac{1}{2}}\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(2x+1)^{\frac{1}{2}}}{\left[(2x+1)^{\frac{1}{2}}\right]^2}$$

$$= \frac{(2x+1)^{\frac{1}{2}}(1) - (x-1)\left(\frac{1}{2}\right)(2x+1)^{-\frac{1}{2}}(2)}{(2x+1)}$$

$$= \frac{(2x+1)^{\frac{1}{2}} - (x-1)(2x+1)^{-\frac{1}{2}}}{(2x+1)}$$

$$= \frac{(2x+1)^{-\frac{1}{2}}\left[(2x+1)-(x-1)\right]}{(2x+1)}$$

$$= \frac{(2x+1)^{-\frac{1}{2}}\left[2x+1-x+1\right]}{(2x+1)}$$

$= (x+2)(2x+1)^{-\frac{3}{2}}$ **(6 marks)**

c. $\dfrac{dy}{dx} = 8\left(x^2+2x+1\right)^7(2x+2)$

$= 8(x+1)^{14}(2)(x+1)$

$= 16(x+1)^{15}$ **(3 marks)**

6. $\dfrac{dy}{dt} = 2$ and $\dfrac{dx}{dt} = 2t-1$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$= \dfrac{2}{2t-1}$ **(4 marks)**

7. $\dfrac{dy}{dt} = \dfrac{dy}{dx} \bullet \dfrac{dx}{dt}$

$= 7(t-1)$

$= 7t-7$ **(2 marks)**

8. $f'(x) = 12x^3 - 5(1) + 0$
$= 12x^3 - 5$

$f''(x) = 12\left(3x^2\right) - 0$
$= 36x^2$

$f''(-1) = 36(-1)^2$
$= 36$ **(3 marks)**

9. a.

$$
\begin{aligned}
2y^2 - xy - x^2 &= 0 \\
2(2yy') - (xy' + y(1)) - 2x &= 0 \\
4yy' - xy' - y - 2x &= 0 \\
(4y - x)y' &= 2x + y \\
y' &= \frac{2x + y}{4y - x}
\end{aligned}
$$

(3 marks)

b. When $x = 1$, $2y^2 - y - 1 = 0$

$$
\begin{aligned}
(2y + 1)(y - 1) &= 0 \\
y &= 1 \text{ or } -\frac{1}{2}
\end{aligned}
$$

At $(1, 1)$, $y' = \frac{2(1) + 1}{4(1) - 1}$
$= 1$

At $\left(1, -\frac{1}{2}\right)$, $y' = \frac{2(1) + \left(-\frac{1}{2}\right)}{4\left(-\frac{1}{2}\right) - 1}$
$= \frac{\frac{3}{2}}{-3}$
$= -\frac{1}{2}$ **(3 marks)**

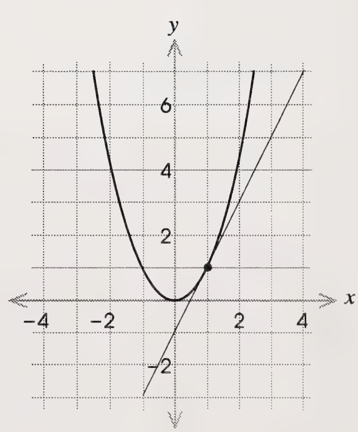
Final Module Assignment Answer Key (20 marks)

1. The slope of the secant through $x = 1$ and $x = 1.1$ is 2.1.

When $x = 1$, $y = 1^2$
$= 1$

When $x = 1.1$, $y = (1.1)^2$
$= 1.21$

Slope of secant $= \frac{1.21 - 1}{1.1 - 1}$
$= 2.1$



The slope of the secant through $x=1$ and $x=1.01$ is 2.01.

When $x=1,\ y=1^2$
$=1$

When $x=1.01,\ y=(1.01)^2$
$=1.0201$

$$\begin{aligned}\text{Slope of secant } &= \frac{1.0201-1}{1.01-1}\\ &= 2.01\end{aligned}$$

The slope of the secant through $x=1$ and $x=1.001$ is 2.001.

When $x=1,\ y=1^2$
$=1$

When $x=1.001,\ y=(1.001)^2$
$=1.002\ 001$

$$\begin{aligned}\text{Slope of secant } &= \frac{1.002\ 001-1}{1.001-1}\\ &= 2.001\end{aligned}$$

The slope of the tangent is 2. **(6 marks)**

2. The derivative is undefined at the following values:

- $x=1$

  The $\lim\limits_{h\to 0^-} \frac{f(1+h)-f(1)}{h}$ does not exist.

- $x=4$

  The tangent line is vertical.

- $x=8$

  $f(8)$ is undefined.

- $x=10$

  The $\lim\limits_{h\to 0^+} \frac{f(10+h)-f(10)}{h}$ does not exist. **(8 marks)**

3. $$\frac{dy}{dx}=x^6(x-1)^{10}\frac{d}{dx}(x+2)^8+(x+2)^8\frac{d}{dx}\left[x^6(x-1)^{10}\right]$$
$$=x^6(x-1)^{10}8(x+2)^7+(x+2)^8\left[x^6\frac{d}{dx}(x-1)^{10}+(x-1)^{10}\frac{d}{dx}\left(x^6\right)\right]$$
$$=8x^6(x-1)^{10}(x+2)^7+(x+2)^8\left[x^6(10)(x-1)^9+(x-1)^{10}(6)x^5\right]$$
$$=8x^6(x-1)^{10}(x+2)^7+10x^6(x+2)^8(x-1)^9+6x^5(x-1)^{10}(x+2)^8$$
$$=2x^5(x-1)^9(x+2)^7[4x(x-1)+5x(x+2)+3(x-1)(x+2)]$$
$$=2x^5(x-1)^9(x+2)^7\left[4x^2-4x+5x^2+10x+3\left(x^2+x-2\right)\right]$$
$$=2x^5(x-1)^9(x+2)^7\left[4x^2-4x+5x^2+10x+3x^2+3x-6\right]$$
$$=2x^5(x-1)^9(x+2)^7\left[12x^2+9x-6\right]$$
$$=6x^5(x-1)^9(x+2)^7\left[4x^2+3x-2\right]$$ **(6 marks)**

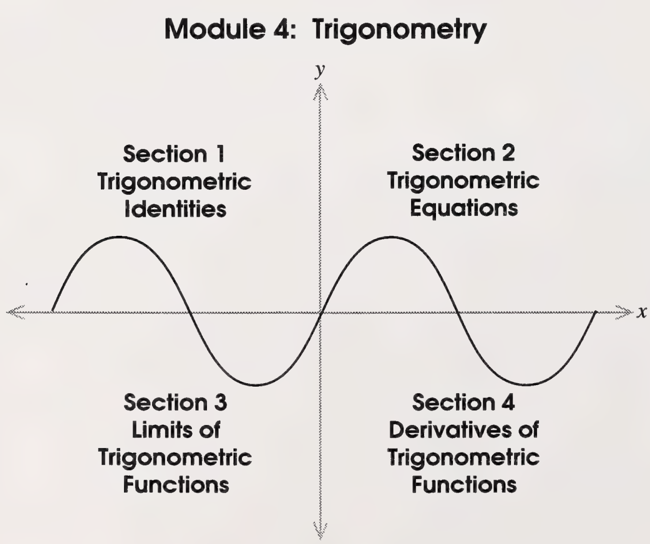
Module 4: Trigonometry

Overview

In Module 3, the student found the derivatives of algebraic functions. This module does the same for trigonometric functions.

Before getting into the derivatives of trigonometric functions, the first two sections reaquaint the student with various trigonometric identities and the solution of trigonometric equations. Students begin by proving more complex trigonometric identities, given some fundamental identities. They then move on to solving trigonometric equations in specified and unspecified intervals.

With this background, students can then move into the concept of derivatives, looking first at the limits of trigonometric expressions as the definition of a derivative. The derivatives of the six trigonometric functions are determined and then applied to more complex trigonometric expressions. Finally, students study one of the uses of the derivative—finding slopes and equations of tangent lines to trigonometric curves. Further applications of the derivative will be studied in Module 6.



Evaluation

The evaluation of this module will be based on four section assignments and one final module assignment.

Section 1 Assignment	**15 marks**
Section 2 Assignment	**13 marks**
Section 3 Assignment	**16 marks**
Section 4 Assignment	**20 marks**
Final Module Assignment	**36 marks**
TOTAL	**100 marks**

Section 1: Trigonometric Identities

Key Concepts

- reciprocal, quotient, and Pythagorean identities
- sum and difference formulas
- double-angle and half-angle formulas

The basic goals of this section are to ensure that students

- simplify trigonometric expressions using the fundamental trigonometric identities
- prove that two trigonometric expressions are equivalent by using trigonometric identities and algebraic manipulations to transform them

Section 1: Assignment Answer Key (15 marks)

1.

LS	RS
$\sec^2 x - 1$	$\sec^2 x \sin^2 x$
$= \tan^2 x$ (Pythagorean identity)	$= \dfrac{1}{\cos^2 x} \bullet \sin^2 x$ (reciprocal identity)
$= \dfrac{\sin^2 x}{\cos^2 x}$ (quotient identity)	$= \dfrac{\sin^2 x}{\cos^2 x}$

LS = RS **(3 marks)**

2. Work with the left side of the equation.

$$\begin{aligned}\frac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x} &= \frac{1-\left(1-2\sin^2 x\right)+2\sin x\cos x}{1+\left(2\cos^2 x-1\right)+2\sin x\cos x} && \text{(double-angle identities)}\\ &= \frac{2\sin^2 x+2\sin x\cos x}{2\cos^2 x+2\sin x\cos x}\\ &= \frac{\cancel{2}\sin x\left(\cancel{\sin x+\cos x}\right)}{\cancel{2}\cos x\left(\cancel{\cos x+\sin x}\right)} && \text{(Factor and cancel.)}\\ &= \frac{\sin x}{\cos x}\\ &= \tan x\end{aligned}$$

Therefore, $\text{LS} = \text{RS}$. **(3 marks)**

3. Work with the left side of the equation.

$$\begin{aligned}\frac{\sin^4\theta-\cos^4\theta}{\sin^2\theta-\cos^2\theta} &= \frac{\left(\sin^2\theta+\cos^2\theta\right)\left(\cancel{\sin^2\theta-\cos^2\theta}\right)}{\cancel{\sin^2\theta-\cos^2\theta}} && \text{(Factor the numerator and cancel.)}\\ &= \sin^2\theta+\cos^2\theta\\ &= 1 && \text{(Pythagorean identity)}\end{aligned}$$

Therefore, $\text{LS} = \text{RS}$. **(2 marks)**

4. Work with the left side of the equation.

$$\begin{aligned}1+2\sin(-x)\cos\left(\frac{\pi}{2}-x\right) &= 1-2\sin x\cos\left(\frac{\pi}{2}-x\right) && \text{(CAST rule)}\\ &= 1-2\sin x\bullet\sin x && \text{(cofunction identity)}\\ &= 1-2\sin^2 x\\ &= \cos 2x && \text{(double-angle identity)}\end{aligned}$$

Therefore, $\text{LS} = \text{RS}$. **(3 marks)**

5. Work with the left side of the equation.

$$\begin{aligned}
\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x} &= \frac{\sin^2 x+(1+\cos x)^2}{\sin x(1+\cos x)} && \text{(common denominator)}\\
&= \frac{\sin^2 x+1+2\cos x+\cos^2 x}{\sin x(1+\cos x)}\\
&= \frac{\sin^2 x+\cos^2 x+1+2\cos x}{\sin x(1+\cos x)} && \text{(Regroup.)}\\
&= \frac{2+2\cos x}{\sin x(1+\cos x)} && \text{(Pythagorean identity)}\\
&= \frac{2(1+\cos x)}{\sin x(1+\cos x)} && \text{(Factor and cancel.)}\\
&= \frac{2}{\sin x}\\
&= 2\csc x && \text{(reciprocal identity)}
\end{aligned}$$

Therefore, $\text{LS}=\text{RS}$. **(4 marks)**

Section 2: Solving Trigonometric Equations

Key Concepts

- solutions within a specified domain
- general solutions when no domain is specified

The basic goals of this section are to ensure that students

- understand that since trionometric functions are periodic, an equation may have more than one solution
- transform trigonometric equations into one or more equations containing a single trigonometric function
- solve equations containing multiple angles
- determine when there are a finite number or infinite number of solutions required

Section 2: Assignment Answer Key (13 marks)

1.

$$\begin{aligned}
\tan^2\theta-\sec\theta-1&=0\\
\left(\sec^2\theta-1\right)-\sec\theta-1&=0 && \text{(Pythagorean identity)}\\
\sec^2\theta-\sec\theta-2&=0\\
(\sec\theta+1)(\sec\theta-2)&=0
\end{aligned}$$

$$\begin{aligned}
\sec\theta+1&=0 & \text{or}\quad \sec\theta-2&=0\\
\sec\theta&=-1 & \sec\theta&=2\\
\cos\theta&=-1 & \cos\theta&=\frac{1}{2}\\
\theta&=\pi & \theta&=\frac{\pi}{3}\text{ and }\frac{5\pi}{3}
\end{aligned}$$

Since there is no specified domain, the general solutions are $\frac{\pi}{3}+2n\pi$, $\pi+2n\pi$, and $\frac{5\pi}{3}+2n\pi$, where n is an integer. **(3 marks)**

2. $\sqrt{3}\cot 3x+1=0$, where $0 \le x \le 2\pi$

$$\sqrt{3}\cot 3x=-1$$

$$\cot 3x=-\frac{1}{\sqrt{3}}$$

$$\tan 3x=-\sqrt{3}$$

$$3x=\frac{2\pi}{3} \text{ and } \frac{5\pi}{3}$$

Since the equation contains a triple angle the domain becomes $0 \le x \le 6\pi$.

$$\therefore\ 3x=\frac{2\pi}{3}, \frac{5\pi}{3}, \frac{2\pi}{3}+2\pi, \frac{5\pi}{3}+2\pi, \frac{2\pi}{3}+4\pi, \frac{5\pi}{3}+4\pi$$

$$3x=\frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}, \frac{11\pi}{3}, \frac{14\pi}{3}, \text{ and } \frac{17\pi}{3}$$

$$x=\frac{2\pi}{9}, \frac{5\pi}{9}, \frac{8\pi}{9}, \frac{11\pi}{9}, \frac{14\pi}{9}, \text{ and } \frac{17\pi}{9}$$

The solutions within one specified domain are $\frac{2\pi}{9}$, $\frac{5\pi}{9}$, $\frac{8\pi}{9}$, $\frac{11\pi}{9}$, $\frac{14\pi}{9}$, and $\frac{17\pi}{9}$. **(4 marks)**

3. $\sin^2 x-\frac{1}{4}=0$, where $-\pi \le x \le \pi$

$$\left(\sin x-\frac{1}{2}\right)\left(\sin x+\frac{1}{2}\right)=0$$

$\sin x-\frac{1}{2}=0$ or $\sin x+\frac{1}{2}=0$

$\sin x=\frac{1}{2}$ $\qquad$ $\sin x=-\frac{1}{2}$

$x=\frac{\pi}{6}$ and $\frac{5\pi}{6}$ $\qquad$ $x=-\frac{5\pi}{6}$ and $-\frac{\pi}{6}$

The solutions within the specified domain are $-\frac{5\pi}{6}$, $-\frac{\pi}{6}$, $\frac{5\pi}{6}$, and $\frac{\pi}{6}$. **(3 marks)**

4. $4\cos^2 x-3=0$, where $0 \le x \le 2\pi$

$$\cos^2 x=\frac{3}{4}$$

$$\cos x=\pm\frac{\sqrt{3}}{2}$$

$$x=\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \text{ and } \frac{11\pi}{6}$$

The solutions within the domain are $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$. **(3 marks)**

Section 3: Limits of Trigonometric Functions

Key Concepts

- the limits of sine and cosine functions
- the limits of more complex trigonometric functions

The basic goals of this section are to ensure that students

- determine the limits of simple trigonometric functions
- determine two special trigonometric limits
- determine the limits of more complex trigonometric functions by transforming them to model fundamental limits

Section 3: Assignment Answer Key (16 marks)

1. a.

$$\begin{aligned}\lim_{x\to 0} x\cot x &= \lim_{x\to 0} x\bullet\frac{\cos x}{\sin x}\\ &= \lim_{x\to 0}\frac{x}{\sin x}\bullet\lim_{x\to 0}\cos x\\ &= 1(1)\\ &= 1 \quad \textbf{(2 marks)}\end{aligned}$$

b.

$$\begin{aligned}\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{\sin x} &= \lim_{x\to 0}\frac{\sin\frac{x}{2}}{\sin x}\bullet\lim_{x\to 0}\sin\frac{x}{2}\\ &= \lim_{x\to 0}\left[\frac{\frac{\frac{x}{2}\bullet\left(\sin\frac{x}{2}\right)}{\frac{x}{2}}}{x\bullet\frac{\sin x}{x}}\right]\bullet\lim_{x\to 0}\sin\frac{x}{2}\\ &= \frac{1}{2}\left[\frac{\lim\limits_{\frac{x}{2}\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}}{\lim\limits_{x\to 0}\frac{\sin x}{x}}\right]\bullet\lim_{x\to 0}\sin\frac{x}{2}\\ &= \frac{1}{2}\left(\frac{1}{1}\right)\bullet\sin\frac{0}{2}\\ &= \frac{1}{2}(0)\\ &= 0 \quad \textbf{(4 marks)}\end{aligned}$$

c.

$$\begin{aligned}\lim_{x\to\pi}\frac{\sin^2 x}{x-\pi} &= \lim_{x-\pi\to 0}\frac{\sin^2(x-\pi)}{x-\pi}\\ &= \lim_{x-\pi\to 0}\frac{\sin(x-\pi)}{x-\pi}\bullet\lim_{x-\pi\to 0}\sin(x-\pi)\\ &= 1\bullet\sin 0\\ &= 1(0)\\ &= 0 \quad \textbf{(3 marks)}\end{aligned}$$

d. Let $\theta = \frac{1}{x}$

$$\begin{aligned}\lim_{x\to\infty} x\sin\frac{1}{x} &= \lim_{\theta\to 0}\frac{1}{\theta}\bullet\sin\theta\\ &= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\\ &= 1\end{aligned}$$

Therefore, the limit is 1. **(3 marks)**

2. When $f(x)=\sin x$, $\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}=\lim\limits_{h\to 0}\dfrac{\sin(2+h)-\sin 2}{h}$

$$
\begin{aligned}
&=\lim_{h\to 0}\frac{\sin 2\ \cos h+\cos 2\ \sin h-\sin 2}{h}\\
&=\lim_{h\to 0}\frac{\sin 2\ \cos h-\sin 2+\cos 2\ \sin h}{h}\\
&=\lim_{h\to 0}\frac{\sin 2(\cos h-1)+\cos 2\ \sin h}{h}\\
&=\lim_{h\to 0}\sin 2\bullet\frac{\cos h-1}{h}+\lim_{h\to 0}\cos 2\bullet\frac{\sin h}{h}\\
&=\sin 2\bullet\lim_{h\to 0}\frac{\cos h-1}{h}+\cos 2\bullet\lim_{h\to 0}\frac{\sin h}{h}\\
&=\sin 2\bullet 0+\cos 2\bullet 1\\
&=\cos 2\\
&\doteq -0.416\,146\,836\\
&\doteq -0.42 \quad \textbf{(4 marks)}
\end{aligned}
$$

Section 4: Derivatives of Trigonometric Functions

Key Concepts

- derivatives of the primary trigonometric functions
- derivatives of the reciprocal functions
- differentiation of complex trigonometric expressions
- trigonometric curves and their tangents

The basic goals of this section are to ensure that the student

- use first principles to determine the derivative of the sine function
- use the identities and rules for differentiating to determine the derivatives of the other five trigonometric functions
- apply basic derivatives and the rules for differentiating to find the derivatives of complex trigonometric expressions
- use implicit differentiation with trigonometric functions
- use the concept of the derivative to determine the slope and equation of a line tangent to a trigonometric curve at a given point

Section 4: Assignment Answer Key (20 marks)

1. a. $y=-3x\cos x$

$$
\begin{aligned}
\frac{dy}{dx}&=-3x\bullet\frac{d}{dx}(\cos x)+\cos x\bullet\frac{d}{dx}(-3x)\\
&=-3x(-\sin x)+(\cos x)(-3)\\
&=3x\sin x-3\cos x \quad \textbf{(2 marks)}
\end{aligned}
$$

b. $y=2\csc^3\left(\sqrt{x}\right)$

$$
\begin{aligned}
\frac{dy}{dx}&=2\bullet\frac{d}{dx}\left(\csc\sqrt{x}\right)^3\\
&=2\bullet 3\csc^2\sqrt{x}\bullet\frac{d}{dx}\left(\csc\sqrt{x}\right)\\
&=6\csc^2\sqrt{x}\left(-\csc\sqrt{x}\ \cot\sqrt{x}\right)\bullet\frac{d}{dx}\left(\sqrt{x}\right)\\
&=-6\csc^3\sqrt{x}\ \cot\sqrt{x}\bullet\left(\frac{1}{2}\right)x^{-\frac{1}{2}}\\
&=-\frac{3}{\sqrt{x}}\csc^3\sqrt{x}\ \cot\sqrt{x} \quad \textbf{(3 marks)}
\end{aligned}
$$

c. $y=\frac{x}{2}-\frac{\sin 2x}{4}$

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x}{2}\right)-\frac{d}{dx}\left(\frac{\sin 2x}{4}\right)$$
$$=\frac{1}{2}-\frac{1}{4}\cos 2x\bullet\frac{d}{dx}(2x)$$
$$=\frac{1}{2}-\frac{1}{2}\cos 2x$$
$$=\frac{1}{2}(1-\cos 2x)$$
$$=\frac{1}{2}\left[1-\left(1-2\sin^2 x\right)\right] \quad \text{(double-angle identity)}$$
$$=\frac{2\sin^2 x}{2}$$
$$=\sin^2 x \quad \textbf{(3 marks)}$$

2. The answer is C. **(1 mark)**

3. a. $x=\sin y+\cos x$

$$\frac{d}{dx}(x)=\frac{d}{dx}(\sin y)+\frac{d}{dx}(\cos x)$$
$$1=\cos y\bullet\frac{dy}{dx}-\sin x$$
$$1+\sin x=\cos y\bullet\frac{dy}{dx}$$
$$\frac{dy}{dx}=\frac{1+\sin x}{\cos y} \quad \textbf{(2 marks)}$$

b. $xy-y^3=\sin x$

$$\left(x\bullet\frac{dy}{dx}+y\bullet 1\right)-3y^2\bullet\frac{dy}{dx}=\frac{d}{dx}(\sin x)$$
$$x\bullet\frac{dy}{dx}-3y^2\bullet\frac{dy}{dx}+y=\cos x$$
$$\frac{dy}{dx}\left(x-3y^2\right)=\cos x-y$$
$$\frac{dy}{dx}=\frac{\cos x-y}{x-3y^2} \quad \textbf{(3 marks)}$$

4. $y=2\sin x+\cos x$

$$\frac{dy}{dx}=2\cos x-\sin x$$

At $x=4.2$, $\frac{dy}{dx}=2\cos(4.2)-\sin(4.2)$

$$\doteq -0.108\,945\,870$$

The slope of the required tangent is approximately -0.11. **(2 marks)**

5. $y = 2\cos x$

At $x = -\frac{\pi}{2}$, $y = 2\cos\left(-\frac{\pi}{2}\right)$ or $2\cos\left(\frac{3\pi}{2}\right) = 2(0)$

$= 0$ $\qquad\qquad\qquad\qquad = 0$

The point of tangency is $\left(-\frac{\pi}{2}, 0\right)$.

$$\frac{dy}{dx} = -2\sin x$$

$$\begin{aligned} \text{At } x = -\tfrac{\pi}{2},\ m &= -2\sin\left(-\tfrac{\pi}{2}\right) \\ &= -2\sin\left(\tfrac{3\pi}{2}\right) \\ &= -2(-1) \\ &= 2 \end{aligned}$$

Therefore, the equation of the tangent line is as follows:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= 2\left(x + \frac{\pi}{2}\right) \\ y &= 2x + \pi \text{ or } 2x - y + \pi = 0 \quad \textbf{(4 marks)} \end{aligned}$$

Final Module Assignment Answer Key (36 marks)

1. $$\begin{aligned} \cos\left(x + \frac{\pi}{2}\right) &= \cos x \cos\frac{\pi}{2} - \sin x \sin\frac{\pi}{2} \\ &= \cos x \bullet 0 - \sin x \bullet 1 \\ &= -\sin x \quad \textbf{(2 marks)} \end{aligned}$$

2. Other methods may be used.

LS	RS
$\dfrac{\tan x+1}{\tan x-1}$	$\dfrac{\sec x+\csc x}{\sec x-\csc x}$
$=\dfrac{\frac{\sin x}{\cos x}+1}{\frac{\sin x}{\cos x}-1}$	$=\dfrac{\frac{1}{\cos x}+\frac{1}{\sin x}}{\frac{1}{\cos x}-\frac{1}{\sin x}}$
$=\dfrac{\frac{\sin x+\cos x}{\cos x}}{\frac{\sin x-\cos x}{\cos x}}$	$=\dfrac{\frac{\sin x+\cos x}{\sin x\cos x}}{\frac{\sin x-\cos x}{\sin x\cos x}}$
$=\dfrac{\sin x+\cos x}{\cancel{\cos x}}\bullet\dfrac{\cancel{\cos x}}{\sin x-\cos x}$	$=\dfrac{\sin x+\cos x}{\cancel{\sin x\cos x}}\bullet\dfrac{\cancel{\sin x\cos x}}{\sin x-\cos x}$
$=\dfrac{\sin x+\cos x}{\sin x-\cos x}$	$=\dfrac{\sin x+\cos x}{\sin x-\cos x}$

LS = RS **(4 marks)**

3. Work with the left side of the equation.

$$
\begin{aligned}
(1+\sin x)^2+(1-\sin x)^2 &= \left(1+2\sin x+\sin^2 x\right)+\left(1-2\sin x+\sin^2 x\right)\\
&=2+2\sin^2 x\\
&=2+2\left(1-\cos^2 x\right)\\
&=2+2-2\cos^2 x\\
&=4-2\cos^2 x
\end{aligned}
$$

Therefore, $\text{LS}=\text{RS}$. **(2 marks)**

4. $-1+\tan 2\theta=0$, where $0\le\theta\le 2\pi$

$$\tan 2\theta=1$$

$$2\theta=\frac{\pi}{4}\text{ and }\frac{5\pi}{4}$$

Domain for 2θ is $0\le\theta\le 4\pi$.

$$
\begin{aligned}
\therefore\ 2\theta &= \frac{\pi}{4},\ \frac{5\pi}{4},\ \frac{\pi}{4}+2\pi,\text{ and }\frac{5\pi}{4}+2\pi\\
2\theta &= \frac{\pi}{4},\ \frac{5\pi}{4},\ \frac{9\pi}{4},\text{ and }\frac{13\pi}{4}\\
\theta &= \frac{\pi}{8},\ \frac{5\pi}{8},\ \frac{9\pi}{8},\text{ and }\frac{13\pi}{8}
\end{aligned}
$$

The solutions within the specified domain are $\frac{\pi}{8}$, $\frac{5\pi}{8}$, $\frac{9\pi}{8}$, and $\frac{13\pi}{8}$. **(4 marks)**

5.

$$\sin\ 2x + 3\cos\ x = 0$$
$$2\sin\ x\ \cos\ x + 3\cos\ x = 0$$
$$\cos\ x(2\sin\ x + 3) = 0$$

$\cos\ x = 0$ or $2\sin\ x + 3 = 0$

$x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ $\qquad$ $\sin\ x = -\frac{3}{2}$

Since no solution exists for $\sin\ x = -\frac{3}{2}$, then $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. Therefore, in an unspecified domain, the solutions are $\frac{\pi}{2} + 2n\pi$ and $\frac{3\pi}{2} + 2n\pi$. **(4 marks)**

6.

$$\begin{aligned}
\lim_{h\to 0} \frac{\cos\ (x+h) - \cos\ x}{h} &= \lim_{h\to 0} \frac{\cos\ x\ \cos\ h - \sin\ x\ \sin\ h - \cos\ x}{h} \\
&= \lim_{h\to 0} \frac{\cos\ x(\cos\ h - 1) - \sin\ x\ \sin\ h}{h} \\
&= \lim_{h\to 0} \cos\ x \bullet \frac{\cos\ h - 1}{h} - \lim_{h\to 0} \sin\ x \bullet \frac{\sin\ h}{h} \\
&= \cos\ x \bullet 0 - \sin\ x \bullet 1 \\
&= -\sin\ x
\end{aligned}$$

This limit determines the derivative of the cosine function. **(3 marks)**

7.

$$\begin{aligned}
\lim_{x\to\frac{\pi}{4}} \frac{1 - \tan\ x}{\sin\ x - \cos\ x} &= \lim_{x\to\frac{\pi}{4}} \frac{1 - \frac{\sin\ x}{\cos\ x}}{\sin\ x - \cos\ x} \\
&= \lim_{x\to\frac{\pi}{4}} \frac{\frac{\cos\ x - \sin\ x}{\cos\ x}}{\sin\ x - \cos\ x} \\
&= \lim_{x\to\frac{\pi}{4}} \frac{-1(\cancel{\sin\ x - \cos\ x})}{\cos\ x} \bullet \frac{1}{\cancel{\sin\ x - \cos\ x}} \\
&= \lim_{x\to\frac{\pi}{4}} -\frac{1}{\cos\ x} \\
&= -\frac{1}{\cos\ \frac{\pi}{4}} \\
&= -\frac{1}{\frac{\sqrt{2}}{2}} \\
&= -\frac{2}{\sqrt{2}} \\
&= -\sqrt{2}\ \textbf{(4 marks)}
\end{aligned}$$

8.

$$y = \sin^2 x$$
$$\frac{dy}{dx} = \frac{d}{dx}(\sin x)^2$$
$$= 2\sin x \bullet \frac{d}{dx}(\sin x)$$
$$= 2\sin x \cos x$$
$$= \cos 2x$$

$$y = \sin x^2$$
$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin x^2\right)$$
$$= \cos x^2 \bullet \frac{d}{dx}\left(x^2\right)$$
$$= 2x\cos x^2$$

The chain rule is used to differentiate both functions; but the exponent in each function has a different base, therefore generating different derivatives. **(4 marks)**

9.

$$\sin 2x = 2\sin x \cos x$$
$$\frac{d}{dx}(\sin 2x) = 2 \bullet \frac{d}{dx}(\sin x \cos x)$$
$$\cos 2x \bullet \frac{d}{dx}(2x) = 2\sin x \bullet \frac{d}{dx}(\cos x) + \cos x \bullet \frac{d}{dx}(\sin x)$$
$$2\cos 2x = 2(\sin x \bullet -\sin x + \cos x \bullet \cos x)$$
$$\cos 2x = \cos^2 x - \sin^2 x$$ (double-angle identity) **(4 marks)**

10.

LS	RS
$y = \frac{1-\cos x}{\sin x}$	$y = \csc x - \cot x$
$\frac{dy}{dx} = \frac{\sin x \bullet \frac{d}{dx}(1-\cos x) - (1-\cos x) \bullet \frac{d}{dx}(\sin x)}{\sin^2 x}$	$\frac{dy}{dx} = \frac{d}{dx}(\csc x) - \frac{d}{dx}(\cot x)$
$= \frac{\sin x \bullet \sin x - (1-\cos x) \bullet \cos x}{\sin^2 x}$	$= -\csc x \cot x + \csc^2 x$
$= \frac{\sin^2 x + \cos^2 x - \cos x}{\sin^2 x}$	
$= \frac{1-\cos x}{\sin^2 x}$	
$= \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}$	
$= \frac{1}{\sin^2 x} - \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)$	
$= \csc^2 x - \csc x \cot x$	

LS = RS

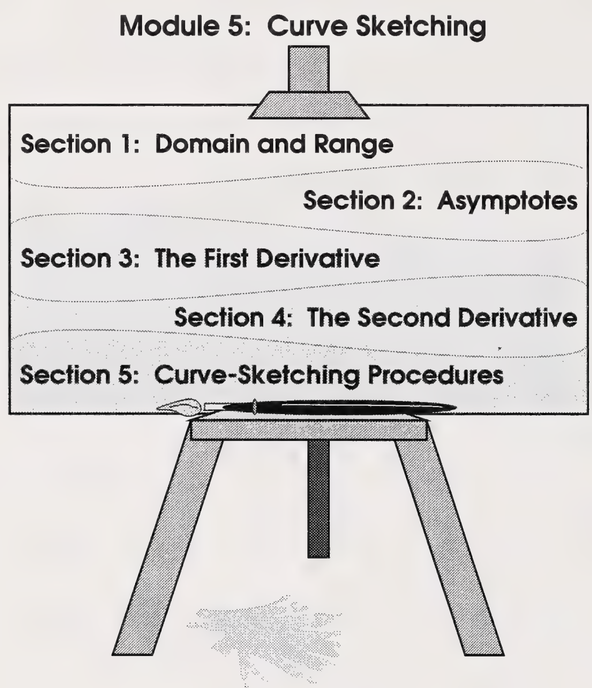
Therefore, the derivatives are equal. **(5 marks)**

Module 5: Curve Sketching

Overview

A fundamental skill in calculus is being able to model real-world situations using functions, and to sketch and interpret the graphs of those functions. This module develops procedures in curve sketching that the student will use not only in subsequent modules, but also in most branches of mathemtaics and the sciences.

Section 1 reviews the concepts of domain and range, and how they can be used to locate the positon of a function's graph in the plane. In Section 2, the student is introduced to techniques for finding horizontal, vertical, or oblique asymptotes. Section 3 extends the application of the first derivative in curve sketching. The student uses it to determine intervals on a graph that are rising and falling, maximum and minimum points, and to locate horizontal and vertical tangents. Section 4 discusses the curvature of a graph. The student describes portions of a curve as either concave upward or concave downward, and locates points of inflection. In Section 5, these strategies are synthesized into an approach used to sketch both algebraic and trigonometric functions.



Evaluation

The evaluation of this module will be based on six assignments:

Section 1 Assignment	**15 marks**
Section 2 Assignment	**15 marks**
Section 3 Assignment	**20 marks**
Section 4 Assignment	**15 marks**
Section 5 Assignment	**30 marks**
Final Module Assignment	**5 marks**
TOTAL	**100 marks**

Mandatory Video

Section 3: Activity 3

Derivatives and Graph Sketching, *Catch 31* series, ACCESS Network, available from the Learning Resources Distributing Centre.

Section 1: Domain and Range

Key Concepts

- domain
- range
- partitioning the plane

The basic goals of this section are to ensure that students

- determine domain and range from tables of values, graphs, and equations of curves
- partition the plane using domain and range

Section 1: Assignment Answer Key (15 marks)

1. The radicand must be non-negative.

$$\begin{aligned} 3x-5 &\geq 0 \\ 3x &\geq 5 \\ x &\geq \frac{5}{3} \end{aligned}$$

The domain is $\left[\frac{5}{3}, \infty\right)$. **(3 marks)**

2.

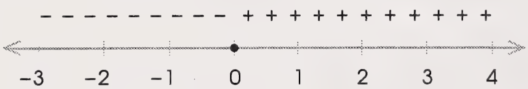
$$\begin{aligned} y &= \frac{4x}{x-3} \\ (x-3)y &= 4x \\ xy-3y &= 4x \\ xy-4x &= 3y \\ x(y-4) &= 3y \\ x &= \frac{3y}{y-4} \end{aligned}$$

Because $y \neq 4$, the range of the function is $(-\infty, 4) \cup (4, \infty)$. **(4 marks)**

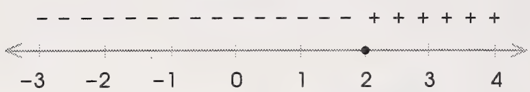
3. The domain is the solution of $x^2 - 2x \geq 0$.

$\therefore\ x(x-2) \geq 0$

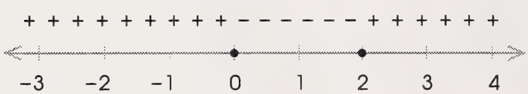
Sign of x

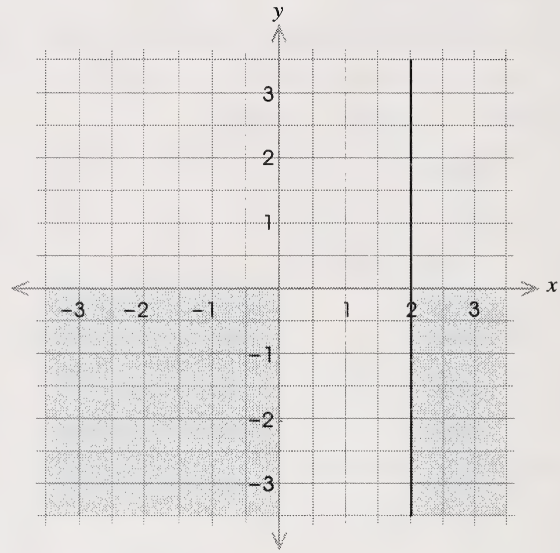


Sign of $x-2$



Sign of $x(x-2)$





Domain $= (-\infty, 0] \cup [2, \infty)$

Since the values of the function are non-positive, the range $= (-\infty, 0]$.

The graph lies in the shaded regions of the plane. **(6 marks)**

4. Domain $= (-1, 3]$

Range $= [-2, 3)$ **(2 marks)**

Section 2: Asymptotes

Key Concepts

- vertical asymptote
- horizontal asymptote
- oblique or slant asymptote

The basic goals of this section are to ensure that students

- define and discuss the conditions under which asymptotes occur
- determine vertical asymptotes of algebraic functions from the zeros of their denominators
- find horizontal asymptotes using limits as $x \to \pm\infty$
- use long division to rewrite functions to locate oblique or slant asymptotes

Section 2: Assignment Answer Key (15 marks)

1. $$\lim_{x\to\infty}\frac{3x+1}{x-2}=\lim_{x\to\infty}\frac{x\left(3+\frac{1}{x}\right)}{x\left(1-\frac{2}{x}\right)}$$
$$=\frac{3+0}{1-0}$$
$$=3$$

The horizontal asymptote is $y=3$. **(3 marks)**

2. **a.** Since the vertical asymptotes are $x=-3$ and $x=4$, then $(x-3)$ and $(x-4)$ are factors of the denominator.

$$\therefore\ x^2+bx+c=(x+3)(x-4)$$
$$=x^2-1x-12$$

Therefore, $b=-1$ and $c=-12$. **(3 marks)**

b. The numerator cannot be of the form $x+3$ or $x-4$. Therefore, $a\neq-3$ and $a\neq4$. **(2 marks)**

3. Use long division.

$$\begin{array}{r|l} & x+1 \\ \hline x^2-x & x^3 \\ & \underline{x^3-x^2} \\ & x^2 \\ & \underline{x^2-x} \\ & x \end{array}$$

$$\therefore\ f(x)=x+1+\frac{x}{x^2-x}$$

As $x\to\infty$, the function approaches the line $y=x+1$. **(4 marks)**

4. As $x\to\infty$, the function $f(x)=2^{-x}+3$ approaches $y=3$. **(2 marks)**

5. Rational functions cannot have both horizontal and oblique asymptotes. **(1 mark)**

Section 3: The First Derivative

Key Concepts

- increasing and decreasing functions
- extrema
- relative (or local) minima and maxima
- absolute maxima and minima
- critical values
- stationary points
- turning points
- interval endpoints
- First Derivative Test
- necessary and sufficient conditions

The basic goals of this section are to ensure that students

- identify, from a graph, locations at which the first derivative is zero or undefined
- relate the zeros of the derivative function to the critical points on the original curve
- explain circumstances where maximum and minimum values occur when $f'(x)$ is not zero
- explain why $f'(x)=0$ will not necessarily yield a maximum or minimum
- verify whether a point is a maximum or minimum
- use the first derivative to find maximum and minimum points to aid in sketching graphs, and comparing these sketches to calculator- or computer-generated plots of the same function
- explain the differences between local and absolute maxima and minima
- explain how the sign of the first derivative indicates whether a curve is rising or falling
- verify whether a point is a maximum or minimum

Section 3: Assignment Answer Key (20 marks)

1. **a.** The function decreases on the interval $(-4, 2)$. **(1 mark)**

b. $f'(-3.9)<0$ and $f'(-4.1)>0$ **(2 marks)**

c. $f(1.9)>f(2)$ and $f(2.1)>f(2)$ **(2 marks)**

d. The critical values are $x=-4$ and $x=2$. **(1 mark)**

2. Differentiate.

$$f(x)=\frac{x}{x^2+1}$$

$$\begin{aligned} f'(x)&=\frac{\left(x^2+1\right)(1)-x(2x)}{\left(x^2+1\right)^2} \\ &=\frac{x^2+1-2x^2}{\left(x^2+1\right)^2} \\ &=\frac{1-x^2}{\left(x^2+1\right)^2} \end{aligned}$$

The curve rises when $1-x^2>0$

$$\therefore\ x^2<1$$
$$-1<x<1 \quad \textbf{(4 marks)}$$

3. $f(x)=x^2-8x+9$, where $-1\le x\le 5$

Use the first derivative to determine critical values.

$$f'(x)=2x-8$$

Critical values occur, in this instance, when $f'(x)=0$.

$$2x-8=0$$
$$x=4$$

$$\begin{aligned}\therefore f(4)&=(4)^2-8(4)+9\\&=16-32+9\\&=-7\end{aligned}$$

To determine whether this is a local minimum or maximum, test to see if the function is increasing or decreasing on either side.

Since $f'(3.5)=-1$, the function falls on the left of $(4,-7)$.

Since $f'(4.5)=1$, the graph rises on the right of $(4,-7)$.

Therefore, $f(4)=-7$ is a relative minimum.

The endpoints of the interval must be tested.

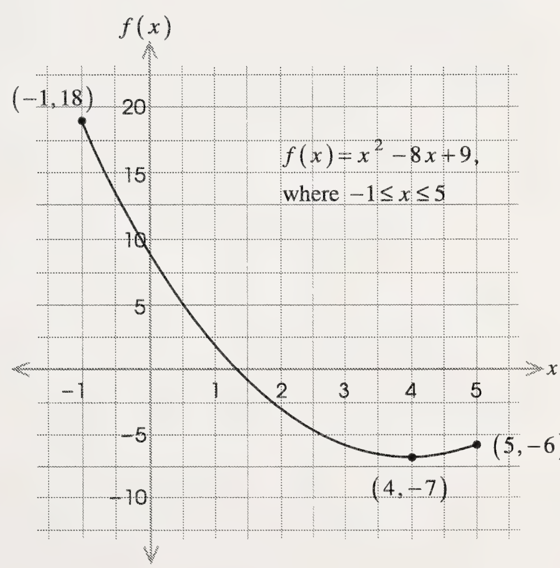
$$\begin{aligned}f(-1)&=(-1)^2-8(1)+9\\&=18\end{aligned}$$

This is a relative maximum since the curve falls to the right of $(-1,18)$.

$$\begin{aligned}f(5)&=(5)^2-8(5)+9\\&=-6\end{aligned}$$

This is a relative maximum since the curve rises to this point.

Therefore, $f(4)=-7$ is an absolute minimum, and $f(-1)=18$ is an absolute maximum. **(8 marks)**



4. Answers will vary.

$f(x)=(x-3)^3$ has a stationary point at $x=3$, but it is not an extreme value. **(2 marks)**

Section 4: The Second Derivative

Key Concepts

- concavity
- Second Derivative Test
- points of inflection

The basic goals of this section are to ensure that students

- explain how the sign of the second derivative indicates the concavity of a graph
- use the first and second derivatives to find maxima, minima, and inflection points to aid in sketching graphs
- identify, from a graph, locations at which the first and second derivatives are zero
- illustrate by examples, that a second derivative of zero is only one possible condition for an inflection point to occur

Section 4: Assignment Answer Key (15 marks)

1. $f(x)=x^2(1-x)=x^2-x^3$

Find the second derivative.

$$f'(x)=2x-3x^2$$
$$f''(x)=2-6x$$

The curve is concave upward when $f''(x)>0$.

$$\therefore\ 2-6x>0$$
$$-6x>-2$$
$$x<\frac{1}{3}$$ **(3 marks)**

2. $f(x)=x^3(4-x)$
$=4x^3-x^4$

Show that a stationary point occurs at $x=3$.

$$f'(x)=12x^2-4x^3$$
$$=4x^2(3-x)$$

Since $f'(3)=0$, a stationary point occurs at $x=3$.

Now, $f(3) = 3^3(4-3)$
$= 27$

Next, find the second derivative.

$$f''(x) = 24x - 12x^2$$

$$\begin{aligned} f''(3) &= 24(3) - 12(3)^2 \\ &= 72 - 108 \\ &< 0 \end{aligned}$$

The curve is concave downward; therefore, $f(3) = 27$ is a relative maximum. **(4 marks)**

3. $y = 5x^3 + 4x^{\frac{5}{2}}$

Find the second derivative.

$$\begin{aligned} y' &= 15x^2 + 4\left(\frac{5}{2}\right)x^{\frac{3}{2}} \\ &= 15x^2 + 10x^{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} y'' &= 30x + 10\left(\frac{3}{2}\right)x^{\frac{1}{2}} \\ &= 30x - 30x^{\frac{1}{2}} \end{aligned}$$

The point of inflection may occur when $y'' = 0$.

$$\begin{aligned} 30x - 30x^{\frac{1}{2}} &= 0 \\ 30x &= 30x^{\frac{1}{2}} \\ x &= x^{\frac{1}{2}} \\ x^2 &= x \\ x^2 - x &= 0 \\ x(x-1) &= 0 \end{aligned}$$

$x = 0$ or $x - 1 = 0$
$x = 1$

Now, $x = 0$ does not give you a point of inflection since the endpoint of the graph occurs at $x = 0$.

$$\begin{aligned} f(1) &= 5(1)^3 + 4(1)^{\frac{5}{2}} \\ &= 9 \end{aligned}$$

Test the concavity on either side.

$$\text{When } x=\tfrac{1}{4},\ f''\left(\tfrac{1}{4}\right)=30\left(\tfrac{1}{4}\right)-30\left(\tfrac{1}{4}\right)^{\frac{1}{2}}$$
$$=7.5-15$$
$$<0$$

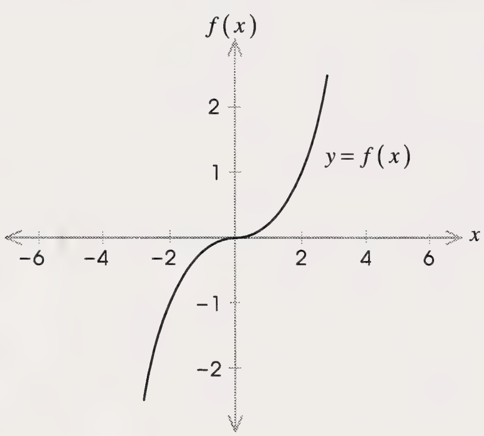
The curve is concave down.

$$\text{When } x=4,\ f''(4)=30(4)-30(4)^{\frac{1}{2}}$$
$$=120-60$$
$$>0$$

The curve is concave upward.

Therefore, $(1, 9)$ is a point of inflection. **(6 marks)**

4. Answers will vary.



(2 marks)

Section 5: Curve Sketching Procedures

Key Concept

systematic approach

The basic goal of this section is to ensure that students employ a systematic calculus procedure to sketch algebraic and trigonometric functions

Section 5: Assignment Answer Key (30 marks)

1. a. The domain is $x \geq 0$ or $[0, \infty)$. **(1 mark)**

b. To find the x-intercepts, equate the function to zero.

$$x^2 - 4\sqrt{x} = 0$$
$$x^2 = 4\sqrt{x}$$
$$x^4 - 16x = 0$$
$$x\left(x^3 - 16\right) = 0$$

$$x = 0 \text{ or } x = 16^{\frac{1}{3}}$$
$$= 2\sqrt[3]{2}$$

The x-intercepts are 0 and $2\sqrt[3]{2}$. Since $f(0) = 0$, the y-intercept is 0. **(3 marks)**

c. No, the graph is not symmetric about the y-axis; the function is only defined for $x \geq 0$.

Test to determine if the graph is symmetric about the x-axis by replacing (x, y) by $(x, -y)$.

$$-y = x^2 - 4\sqrt{x}$$
$$y = -x^2 + 4\sqrt{x}$$

This is not the same as $y = x^2 - 4\sqrt{x}$. The graph is not symmetric about the x-axis. **(2 marks)**

d. There are no asymptotes. **(1 mark)**

e. Differentiate.

$$f'(x) = 2x - 2x^{-\frac{1}{2}}$$

The function increases when $f'(x) > 0$ and decreases when $f'(x) < 0$.

$$\therefore\ 2x - 2x^{-\frac{1}{2}} > 0$$
$$2x > 2x^{-\frac{1}{2}}$$
$$x > x^{-\frac{1}{2}}$$
$$x^2 > x^{-1}$$
$$x^2 > \frac{1}{x}$$
$$x^3 > 1 \quad (\text{since } x > 0)$$
$$x > 1$$

$$\therefore\ 2x - 2x^{-\frac{1}{2}} < 0$$
$$2x < 2x^{-\frac{1}{2}}$$
$$x < x^{-\frac{1}{2}}$$
$$x^2 < x^{-1}$$
$$x^2 < \frac{1}{x}$$
$$x^3 < 1$$
$$0 < x < 1$$

The function increases when $x > 1$, and the function decreases when $0 < x < 1$. **(3 marks)**

f. Extreme values occur at $x = 0$, the endpoint of the domain, and at $x = 1$ (where a stationary point occurs).

$$f(0) = 0$$

This is a local maximum since the function decreases for $0 < x < 1$.

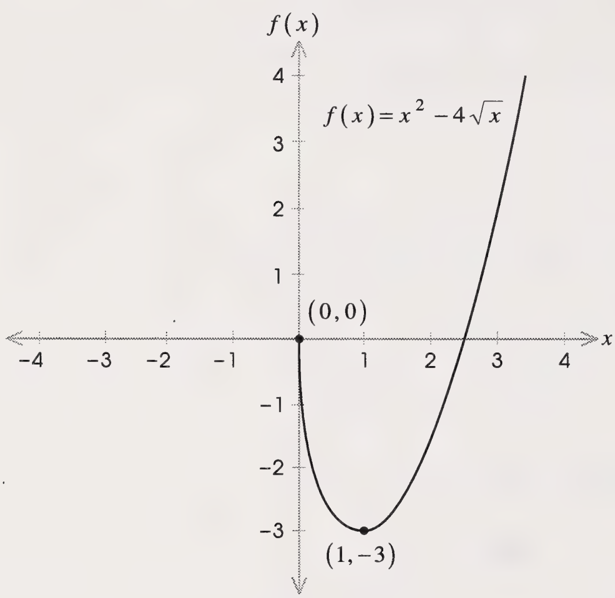
$$\begin{aligned} f(1) &= 1^2 - 4\sqrt{1} \\ &= -3 \end{aligned}$$

This is a local minimum since the function decreases for $0 < x < 1$ and increases for $x > 1$. **(3 marks)**

g. Since $f''(x) = 2 + x^{-\frac{3}{2}}$ is positive for all $x > 0$, this function is concave upward for $x > 0$. **(2 marks)**

h. Since the curve does not change concavity, there are no points of inflection **(1 mark)**

i.



(2 marks)

2. **a.** Stationary points occur when $f'(x)=0$.

$$\begin{aligned} f'(x)=\cos x-\sin x &= 0 \\ \cos x &= \sin x \\ \frac{\sin x}{\cos x} &= 1 \\ \tan x &= 1 \end{aligned}$$

In the interval $0 \leq x \leq \pi$, $\tan x=1$ at $x=\frac{\pi}{4}$.

Now, $f\left(\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{4}\right)$

$$\begin{aligned} &= \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2} \\ &= \sqrt{2} \end{aligned}$$

Apply the Second Derivative Test.

$$\begin{aligned} f''(x) &= -\sin x-\cos x \\ f''\left(\frac{\pi}{4}\right) &= -\sin\left(\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{4}\right) \\ &< 0 \end{aligned}$$

Since the curve is concave downward at $\left(\frac{\pi}{4}, \sqrt{2}\right)$, then that point is a relative maximum. **(4 marks)**

b. Points of inflection may occur when $f''(x)=0$.

$$\begin{aligned} -\sin x-\cos x &= 0 \\ \sin x &= -\cos x \\ \frac{\sin x}{\cos x} &= -1 \\ \tan x &= -1 \end{aligned}$$

In the interval $0 \leq x \leq \pi$, $\tan x=-1$ at $x=\frac{3\pi}{4}$.

$$\begin{aligned} \therefore f\left(\frac{3\pi}{4}\right) &= \sin\left(\frac{3\pi}{4}\right)+\cos\left(\frac{3\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2} \\ &= 0 \end{aligned}$$

Check the concavity on either side.

Recall $f''\left(\frac{\pi}{4}\right) < 0$; on the left the curve is concave downward.

$$\begin{aligned}\text{Also, } f''\left(\frac{5\pi}{6}\right) &= -\sin\left(\frac{5\pi}{6}\right) - \cos\left(\frac{5\pi}{6}\right)\\ &= -\frac{1}{2} - \left(\frac{\sqrt{3}}{2}\right)\\ &= -\frac{1}{2} + \frac{\sqrt{3}}{2}\\ &> 0\end{aligned}$$

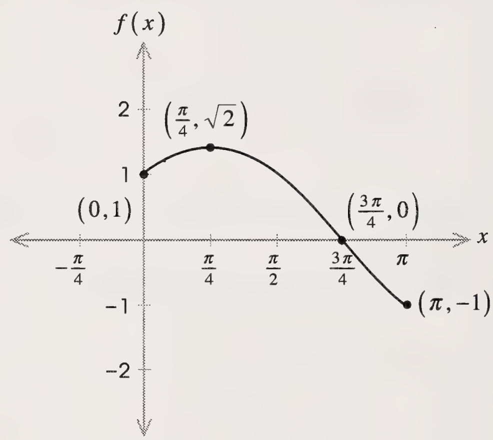
On the right the curve is concave upward.

Therefore, $\left(\frac{3\pi}{4}, 0\right)$ is a point of inflection. **(4 marks)**

c. $f(0) = \sin 0 + \cos 0$
$= 0 + 1$

$f(\pi) = \sin \pi + \cos \pi$
$= 0 - 1$
$= -1$

The endpoints of the interval are $(0, 1)$ and $(\pi, -1)$.
(4 marks)



Final Module Assignment (5 marks)

For questions 1 to 5, the answers will vary. Accept any answer that fits the requirements. The following are examples only.

1. $f(x) = x^2$ **(1 mark)**

2. $f(x) = x^{\frac{1}{3}}$ **(1 mark)**

3. $f(x) = \dfrac{x^3}{x^2 - 1}$ **(1 mark)**

4. $f(x) = x^3$ **(1 mark)**

5. $f(x) = x$ **(1 mark)**

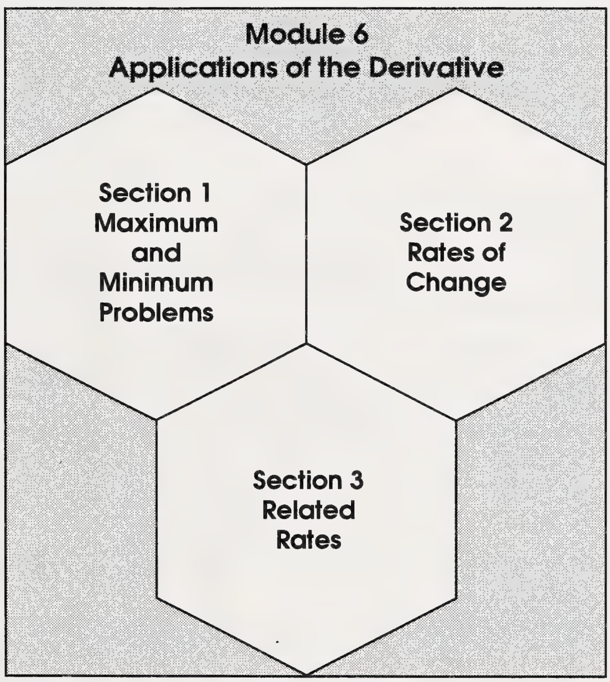
Module 6: Applications of the Derivative

Overview

This module discusses further applications of the derivative. The first application, curve sketching, was studied in Module 5. Now the derivative will be looked at as a means to solve other types of problems.

Section 1 shows how the derivative can be used to solve optimization problems. Section 2 deals with rates of change as derivatives, velocity and acceleration being the most common. Section 3 looks at the rates of change of two or more related variables that are changing with respect to time.

Throughout the module the student will find practical applications for this mathematical concept. The student will see how calculus can be used to solve optimization and rate problems in science, engineering, business, and economics.



Evaluation

The evaluation of this module will be based on four assignments:

Section 1 Assignment	**25 marks**
Section 2 Assignment	**24 marks**
Section 3 Assignment	**21 marks**
Final Module Assignment	**30 marks**
TOTAL	**100 marks**

Mandatory Videos

Section 1: Activity 2

Maxima and Minima, *Catch 31* series, ACCESS Network, available from the Learning Resources Distributing Centre.

Section 2: Activity 2

Motion: Distance, Velocity, and Acceleration, *Catch 31* series, ACCESS Network, available from the Learning Resources Distributing Centre.

Functions Defined Implicitly, *Catch 31* series, ACCESS Network, available from the Learning Resources Distributing Centre.

Section 3: Activity 1

Related Rates, *Catch 31* series, ACCESS Network, available from the Learning Resources Distributing Centre.

Section 1: Maximum and Minimum Problems

Key Concepts

- finding maximum and minimum values in algebra and geometry
- finding extreme values in distances and times
- using maxima and minima to solve problems in economics and other sciences

The basic goals of this section are to ensure that students

- can explain the appropriateness of finding a maximum value or a minimum value
- can find maxima or minima that solve a problem, given a model in the form of an equation

Section 1: Assignment Answer Key (25 marks)

1. Let x be one number and y be the other number.
Let P be the product to be maximized.

$$\begin{aligned} x + y &= 18 \\ x &= 18 - y \end{aligned}$$

$$\begin{aligned} P &= xy^2 \\ &= (18 - y)y^2 \\ &= 18y^2 - y^3 \end{aligned}$$

$$\begin{aligned} \frac{dP}{dy} = 36y - 3y^2 &= 0 \\ 3y(12 - y) &= 0 \end{aligned}$$

$$3y = 0 \quad \text{or} \quad 12 - y = 0$$

$$y = 0 \qquad y = 12 \qquad (y \text{ must be a positive value.})$$

$$\frac{d^2P}{dy^2} = 36 - 6y$$

When $y = 12$, $\frac{d^2P}{dy^2} = 36 - 6(12)$
$= -36$

Therefore, $y = 12$ yields a maximum.

When $y = 12$, $x = 18 - 12$
$= 6$

Therefore, the two numbers are 6 and 12. **(5 marks)**

2. $V = \pi r^2 h$

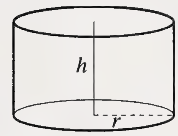
$500 = \pi r^2 h$

$h = \dfrac{500}{\pi r^2}$

$A = 2\pi r^2 + 2\pi rh$

$= 2\pi r^2 + 2\pi r \times \dfrac{500}{\pi r^2}$

$= 2\pi r^2 + \dfrac{1000}{r}$



$$\frac{dA}{dr} = 4\pi r - \frac{1000}{r^2} = 0$$

$$4\pi r^3 = 1000$$

$$r^3 = \frac{1000}{4\pi}$$

$$r^3 = \frac{250}{\pi}$$

$$r \doteq 4.3$$

$$h \doteq \frac{500}{\pi(4.3)^2}$$

$$\doteq 8.6$$

The radius is approximately 4.3 cm and the height is approximately 8.6 cm. **(5 marks)**

3. $V = \frac{1}{3}\pi r^2 h$

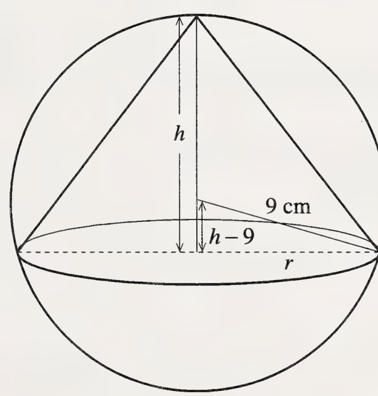
$9^2 = r^2 + (h-9)^2$

$81 = r^2 + h^2 - 18h + 81$

$r^2 = 18h - h^2$

$\therefore\ V = \frac{1}{3}\pi\left(18h - h^2\right)h$

$= \frac{1}{3}\pi\left(18h^2 - h^3\right)$



$$\frac{dV}{dh} = \frac{1}{3}\pi\left(36h - 3h^2\right) = 0$$

$$h(36 - 3h) = 0$$

$h = 0$ or $36 - 3h = 0$ (Discard $h = 0$.)

$h = 12$

$r^2 = 18(12) - (12)^2$

$r^2 = 72$

$r = \sqrt{72}$

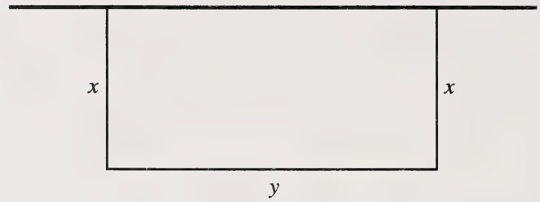
$= 6\sqrt{2}$

$V = \frac{1}{3}\pi(72)(12)$

$= 288\pi$

The maximum volume is $288\pi\,\text{cm}^3$. **(7 marks)**

4. Let x be the width (in m) and y be the length (in m).
Let A be the area to be maximized.



$$2x + y = 30$$
$$y = 30 - 2x$$

$$A = xy$$
$$= x(30 - 2x)$$
$$= 30x - 2x^2$$

$$\frac{dA}{dx} = 30 - 4x = 0$$
$$4x = 30$$
$$x = 7.5$$

When $x = 7.5,\ y = 30 - 2(7.5)$
$$= 15$$

Therefore, the flower plot that maximizes the area is 7.5 m by 15 m. **(4 marks)**

5. Profit = revenue − cost

$$P = \left(700x - 0.25x^2\right) - (500 + 4x)$$
$$= 696x - 0.25x^2 - 500$$

$$\frac{dP}{dx} = 696 - 0.5x = 0$$
$$0.5x = 696$$
$$x = 1392$$

Weekly profit is at a maximum when 1392 toy cars are produced. **(4 marks)**

Section 2: Rates of Change

Key Concepts

- velocity and acceleration as rates of change
- rates of change in the sciences

The basic goals of this section are to ensure that students

- can express the connection between a derivative and the appropriate rate of change
- are able to use this connection to relate complicated rates of change to simpler ones

Section 2: Assignment Answer Key (24 marks)

1. **a.** $v = \frac{ds}{dt} = -4t + 28$ **(1 mark)**

b. When $t = 3,\ v = -4(3) + 28$
$$= 16$$

The velocity at $t = 3$ s is 16 m/s. **(1 mark)**

c. $\frac{ds}{dt} = -4t + 28 = 0$

$4t = 28$

$t = 7$

The object reaches maximum displacement in 7 s. **(1 mark)**

d. When $t = 7$, $s = -2(7)^2 + 28(7) + 45$

$= -98 + 196 + 45$

$= 143$

The maximum displacement is 143 m. **(1 mark)**

e. When $t = 0$, $s = -2(0)^2 + 28(0) + 45$

$= 45$

When $t = 8$, $s = -2(8)^2 + 28(8) + 45$

$= -128 + 224 + 45$

$= 141$

$v_{ave} = \frac{141 - 45}{8 - 0}$

$= 12$

The average velocity from $t = 0$ to $t = 8$ is 12 m/s. **(2 marks)**

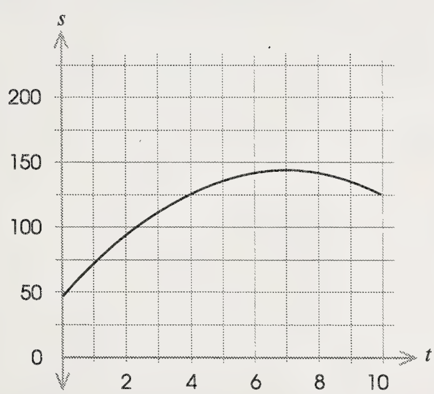
f. $a = \frac{dv}{dt} = -4 \text{ m/s}^2$ **(1 mark)**

g. $a = -4 \text{ m/s}^2$

The acceleration at $t = 4$ s is -4 m/s^2. **(1 mark)**

h.

t	0	1	2	3	4	5	6	7	8	9	10
s	45	71	93	111	125	135	141	143	141	135	125

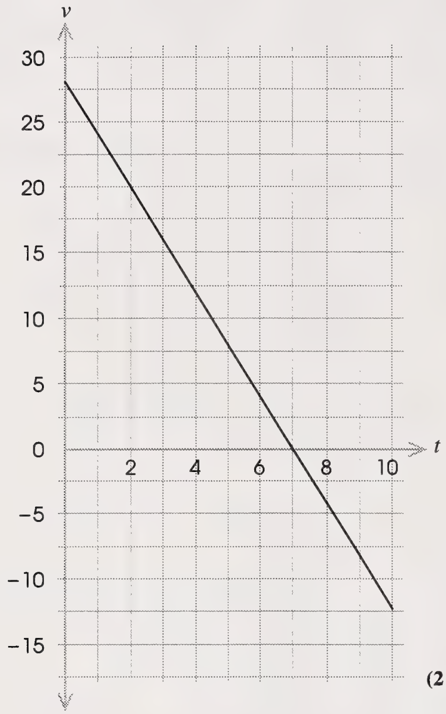


(2 marks)

i. At $t = 3$ s, the direction of the motion is positive.
At $t = 7$ s, the object stops. It has reached its maximum displacement.
At $t = 9$ s, the direction of the motion is negative. The object is moving in the opposite direction. **(2 marks)**

j.

t	v
0	28
1	24
7	0
10	−12



(2 marks)

k. The velocity of the object is decreasing from $t = 0$ s to $t = 7$ s. At $t = 7$ s, the object stops and then moves in the opposite direction. The acceleration is a negative constant. It is actually a constant deceleration until $t = 7$ s. After $t = 7$ s, it accelerates but in the opposite direction; therefore, the acceleration is always -4 m/s^2. **(2 marks)**

2.

$$v = \frac{ds}{dt} = -6t^2 - 8t + 6$$

$$-6t^2 - 8t + 6 = -2$$

$$-6t^2 - 8t + 8 = 0$$

$$3t^2 + 4t - 4 = 0$$

$$(3t - 2)(t + 2) = 0$$

$3t - 2 = 0$ or $t + 2 = 0$

$t = \frac{2}{3}$ $\qquad t = -2$ $\qquad$ (t cannot be negative.)

$$a = \frac{dv}{dt} = -12t - 8$$

When $t = \frac{2}{3}$, $a = -12\left(\frac{2}{3}\right) - 8$

$$= -8 - 8$$

$$= -16$$

The acceleration is -16 m/s^2. **(4 marks)**

3. $\frac{d}{dt}\left(\frac{s^2}{3}\right)-\frac{d}{dt}\left(4v^2\right)=\frac{d}{dt}(12)$

$$\frac{2}{3}s\frac{ds}{dt}-8v\frac{dv}{dt}=0$$

$$\frac{2s}{3}(v)-8v(a)=0$$

$$a=\frac{2vs}{3}\div(8v)$$

$$=\frac{s}{12}$$

The acceleration is $\frac{s}{12}$. **(4 marks)**

Section 3: Related Rates

Key Concepts

- relation of two or more rates, particularly with respect to time
- areas and volumes that change with time
- the motion of two objects related to time
- trigonometric functions and related rates

The basic goals of this section are to ensure that students

- use the chain rule to find the derivative of a function with respect to an external variable
- calculate related rates for the time derivatives of area, volume, surface area, and relative motion
- calculate related rates of change with respect to time in scientific applications
- solve related-rate problems that use models containing trigonometric functions

Section 3: Assignment Answer Key (21 marks)

1. Let the length of each side be x cm and the height be h cm.

$$h^2=x^2-\left(\frac{x}{2}\right)^2$$

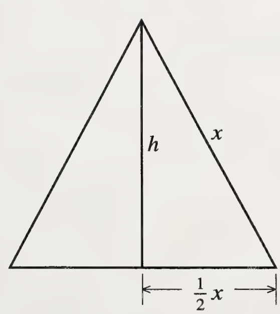
$$h^2=x^2-\frac{x^2}{4}$$

$$h^2=\frac{4x^2-x^2}{4}$$

$$h^2=\frac{3x^2}{4}$$

$$h=\frac{\sqrt{3}x}{2}$$

$$x=\frac{2h}{\sqrt{3}}$$



$$\text{Area } A = \frac{1}{2}xh$$

$$A = \frac{1}{2}\left(\frac{2h}{\sqrt{3}}\right)h$$

$$= \frac{h^2}{\sqrt{3}}$$

$$\frac{dA}{dt} = \frac{2h}{\sqrt{3}} \times \frac{dh}{dt}$$

Since $\frac{dh}{dt} = 3$ cm/min, $\frac{dA}{dt} = \frac{2}{\sqrt{3}}(5)(3)$

$$= \frac{30}{\sqrt{3}}$$

$$= 10\sqrt{3}$$

$$\doteq 17.32$$

The area of the triangle is changing at about $17.32\ \text{cm}^2/\text{min}$. **(4 marks)**

2. Let the radius of the water's surface be r cm and its depth be h cm.

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{r}{8} = \frac{h}{10}$$

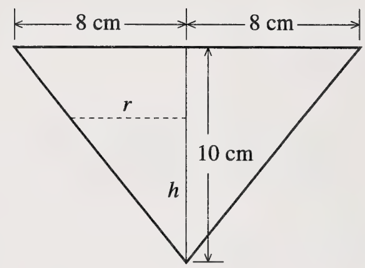
$$r = \frac{8h}{10}$$

$$= \frac{4h}{5}$$

$$V = \frac{1}{3}\pi\left(\frac{16h^2}{25}\right)h$$

$$= \frac{16\pi}{75}h^3$$

$$\frac{dV}{dt} = \frac{16\pi}{75}\left(3h^2\right)\frac{dh}{dt}$$



When $h = 5$ cm, $\frac{dV}{dt} = -25\ \text{cm}^3/\text{s}$.

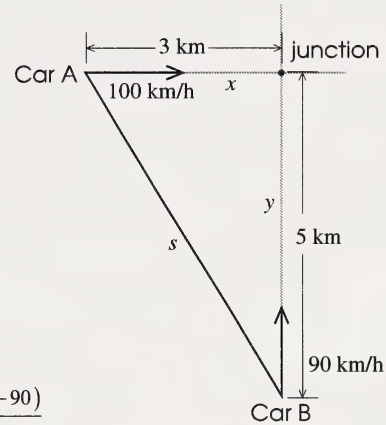
$$\therefore \frac{dh}{dt} = \frac{(-25)(75)}{16\pi(3)(25)}$$

$$= -\frac{25}{16\pi}$$

$$\doteq -0.497$$

The depth of the water is decreasing at about 0.497 cm/s. **(5 marks)**

3. Let x be the distance Car A is from the intersection.
Let y be the distance Car B is from the intersection.



$$s^2 = x^2 + y^2$$

$$\frac{d}{dt}\left(s^2\right) = \frac{d}{dt}\left(x^2\right) + \frac{d}{dt}\left(y^2\right)$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{x \bullet \frac{dx}{dt} + y \bullet \frac{dy}{dt}}{s}$$

$\frac{dx}{dt} = -100$ km/h, $\frac{dy}{dt} = -90$ km/h, $x = 3$ km, and $y = 5$ km

$$\therefore\ s = \sqrt{3^2 + 5^2} = \sqrt{34}$$

$$\therefore\ \frac{ds}{dt} = \frac{3(-100)+5(-90)}{\sqrt{34}} = \frac{-750}{\sqrt{34}} \doteq -128.62$$

The distance between the two cars is decreasing at a rate of about 128.62 km/h. **(4 marks)**

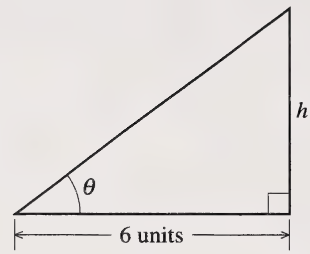
4. $\frac{dy}{dt} = -6x\frac{dx}{dt} + 2\frac{dx}{dt}$

When $x = 5$, $\frac{dx}{dt} = 2$.

$$\therefore\ \frac{dy}{dt} = -6(5)(2)+2(2) = -60+4 = -56$$

The y-coordinate is changing at -56 units/s. **(3 marks)**

5. Given $\frac{dh}{dt} = 1$ unit/s, find $\frac{d\theta}{dt}$ when $h = 8$.



$$\tan\theta = \frac{h}{6}$$

$$\sec^2\theta \bullet \frac{d\theta}{dt} = \frac{1}{6} \bullet \frac{dh}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{6} \bullet \frac{dh}{dt} \bullet \cos^2\theta$$

$$\frac{1}{\sec\theta} = \cos\theta$$

When $h = 8$,
$$\cos\theta = \frac{8}{\sqrt{6^2 + 8^2}} = \frac{8}{10} = \frac{4}{5}$$

$$\therefore \frac{d\theta}{dt} = \frac{1}{6}(1)\left(\frac{4}{5}\right)^2 = \frac{8}{75}$$

The angle is changing at a rate of $\frac{8}{75}$ or about 0.107 rad/s. **(5 marks)**

Final Module Assignment Answer Key (30 marks)

1. Let x be one number and y be the other number.
   Let S be the sum to be minimized.

$$x + y = 8$$
$$x = 8 - y$$

$$S = x^2 + y^3$$
$$= (8 - y)^2 + y^3$$
$$= 64 - 16y + y^2 + y^3$$

$$\frac{dS}{dy} = -16 + 2y + 3y^2 = 0$$
$$3y^2 + 2y - 16 = 0$$
$$(3y + 8)(y - 2) = 0$$

$3y + 8 = 0$ or $y - 2 = 0$

$y = -\frac{8}{3}$ $\quad$ $y = 2$ $\quad$ (y cannot be negative.)

$\frac{d^2S}{dy^2} = 6y$ (A positive value indicates a minimum.)

Therefore, if $y = 2$, then $x = 6$.

The numbers are 6 and 2. **(4 marks)**

2. Let x be the price increase (in \$), $(16+x)$ be the adjusted price per ticket (in \$), and $(4000-100x)$ be the number of tickets sold at the adjusted price.

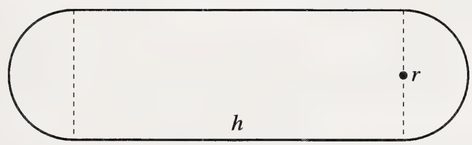
$$R(x) = \text{number of tickets} \times \text{price per ticket}$$
$$\begin{aligned} R(x) &= (4000-100x)(16+x) \\ &= 64\,000 + 2400x - 100x^2 \end{aligned}$$

$$\begin{aligned} \frac{dR}{dx} = 2400 - 200x &= 0 \\ 200x &= 2400 \\ x &= 12 \end{aligned}$$

$$\frac{d^2R}{dx^2} = -200 \quad \text{(A negative value indicates a maximum.)}$$

The maximum revenue is realized when tickets are priced at \$28. **(4 marks)**

3. Let r be the radius of the cylinder and each hemisphere.
Let SA be the surface area to be minimized.
Let h be the length of the cylindrical portion.



$$V = \frac{4\pi r^3}{3} + \pi r^2 h = 120$$
$$\begin{aligned} h &= \frac{120}{\pi r^2} - \frac{4\pi r^3}{3\pi r^2} \\ &= \frac{120}{\pi r^2} - \frac{4r}{3} \end{aligned}$$

$$\begin{aligned} SA &= 4\pi r^2 + 2\pi r h \\ &= 4\pi r^2 + 2\pi r\left(\frac{120}{\pi r^2} - \frac{4r}{3}\right) \\ &= 4\pi r^2 + \frac{240}{r} - \frac{8\pi r^2}{3} \end{aligned}$$

$$\begin{aligned} \frac{dSA}{dr} = 8\pi r - \frac{240}{r^2} - \frac{16\pi r}{3} &= 0 \\ \frac{24\pi r}{3} - \frac{240}{r^2} - \frac{16\pi r}{3} &= 0 \\ \frac{8\pi r}{3} - \frac{240}{r^2} &= 0 \\ \frac{8\pi r}{3} &= \frac{240}{r^2} \\ r^3 &= 240 \bullet \frac{3}{8\pi} \\ &= \frac{90}{\pi} \\ t &= \sqrt[3]{\frac{90}{\pi}} \\ &\doteq 3.06 \end{aligned}$$

The radius that will mimimize the surface area is about 3.06 cm. **(5 marks)**

4. $a=\frac{dv}{dt}=\frac{(3+s)\frac{d}{dt}(150s)-(150s)\frac{d}{dt}(3+s)}{(3+s)^2}$

$$=\frac{(3+s)(150)v-(150s)v}{(3+s)^2}$$

$$=\frac{450v+150vs-150vs}{(3+s)^2}$$

$$=\frac{450v}{(3+s)^2}$$

$$=\frac{450\left(\frac{150s}{3+s}\right)}{(3+s)^2}$$

The value of v was given.

$$=\frac{67\,500s}{(3+s)^3}$$

The acceleration in terms of s is $\frac{67\,500s}{(3+s)^3}$. **(4 marks)**

5. Given $\frac{dr}{dt}=25$ cm/s, find $\frac{dA}{dt}$ when $r=100$ cm.

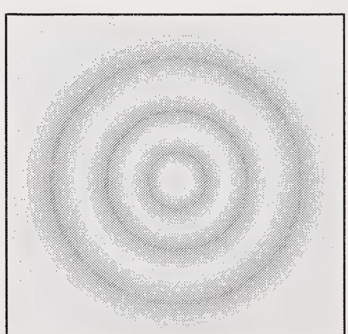
$$A=\pi r^2$$

$$\frac{dA}{dt}=2\pi r\frac{dr}{dt}$$

$$=2\pi(100)(25)$$

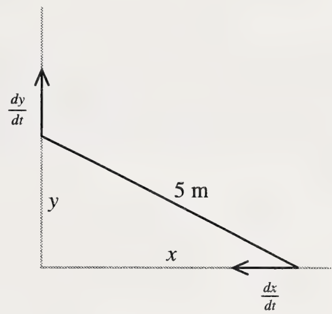
$$=5000\pi$$

The area of the disturbed water is increasing at a rate of 5000π cm/s. **(3 marks)**



6. Let x be the distance from the bottom of the plank to the wall, and let y be the height from the top of the plank to the ground.

Find y when $x = 2$ m.



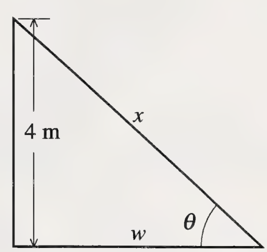
$$x^2 + y^2 = 5^2$$
$$y = \sqrt{25-(2)^2}$$
$$= \sqrt{21}$$

Given $\frac{dy}{dt} = 15 \text{ cm/s} = 0.15 \text{ m/s}$, find $\frac{dx}{dt}$.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$
$$2(2)\frac{dx}{dt} + 2\left(\sqrt{21}\right)(0.15) = 0$$
$$\frac{dx}{dt} = \frac{-2\left(\sqrt{21}\right)(0.15)}{4}$$
$$\doteq -0.344 \text{ m/s or } -34.4 \text{ cm/s}$$

The end of the plank is sliding along the ground at about 34.4 cm/s. **(4 marks)**

7. Given $\frac{dx}{dt} = -30 \text{ cm/s} = -0.3 \text{ m/s}$ (negative because the line is decreasing in length), find $\frac{d\theta}{dt}$ when $x = 8$ m.



$$\csc\theta = \frac{x}{4}$$
$$-\csc\theta \cot\theta \frac{d\theta}{dt} = \left(\frac{1}{4}\right)\frac{dx}{dt}$$
$$\frac{d\theta}{dt} = \left(-\frac{1}{4}\right)\frac{dx}{dt} \bullet \sin\theta \tan\theta$$

When $x = 8$, $w = \sqrt{8^2 - 4^4}$
$$= \sqrt{48}$$
$$= 4\sqrt{3}$$

Therefore, $\sin\theta = \frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$.

$$\frac{d\theta}{dt} = \left(-\frac{1}{4}\right)(-0.3)\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{0.3}{8\sqrt{3}}$$
$$\doteq 0.022$$

The angle is increasing at a rate of about 0.022 rad/s. **(6 marks)**

Module 7: The Integral

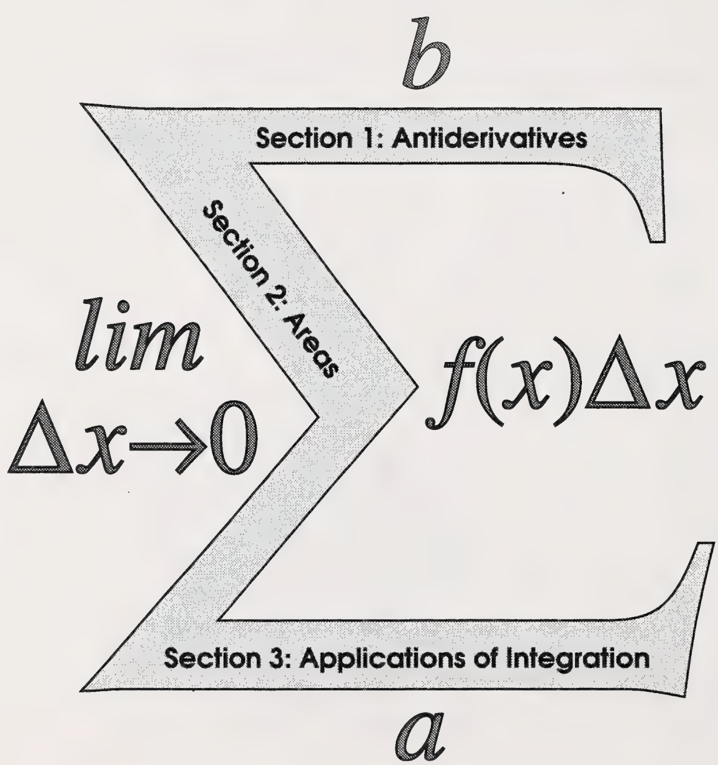
Overview

Module 7 introduces the student to integral calculus. In the preceding modules, the student focussed on the derivative—its procedures and applications. In this module, the student works with the inverse operation of finding antiderivatives or integrals. Given a derivative or a differential equation, the student is expected to solve for the original function or a family of functions with that derivative. Section 1 of this module deals the specifics of this process, covering basic techniques for finding antiderivatives for both algebraic and trigonometric functions.

Section 2 introduces the student to area as a basic application of integral calculus. The student begins evaluating areas between curves and the x-axis by subdividing those areas into thin rectangular strips, and applying limits to determine the sum of the areas of those strips. Next, the Fundamental Theorem of Calculus links this process to the definite integral. Once that link has been made, the student refines techniques for determining areas between curves and the x-axis, and for evaluating areas between curves.

In Section 3, the student applies integral calculus to motion problems—relating position, displacement, velocity, and acceleration. Students use calculus procedures to solve problems in kinematics and harmonic motion. In the final activity, the student determines average values of functions such as average velocities or forces.

Module 7: The Integral



Evaluation

The evaluation of this module will be based on four assignments:

Section 1 Assignment	**36 marks**
Section 2 Assignment	**40 marks**
Section 3 Assignment	**14 marks**
Final Module Assignment	**10 marks**
TOTAL	**100 marks**

Mandatory Videos

Section 2: Activity 3

Integration, Catch 31 series, ACCESS Network, available from the Learning Resources Distributing Centre.

Section 2: Activity 5

Applications of Integrals to Area and Motion, Catch 31 series, ACCESS Network, available from the Learning Resources Distributing Centre.

Section 1: Antiderivatives

Key Concepts

- general antiderivative
- primitive
- indefinite integral
- family of curves
- differential equation
- initial conditions
- comparison techniques

The basic goals of this section are to ensure that students

- find the antiderivatives of polynomials, rational algebraic functions, and simple trigonometric functions
- use comparison techniques to determine indefinite integrals
- identify the indefinite integral $\int f(x)\ dx$ as the sum of an antiderivative $F(x)$ and a constant C
- find the family of curves whose first derivative was given, and graph that family as a sequence of functions on the same grid
- solve first-order differential equations for general and specific solutions
- apply the following general rules for finding integrals

$$\int f(x)\ dx = F(x)+C$$

$$\int af(x)\ dx = a\int f(x)\ dx$$

$$\int [f(x)\pm g(x)]\ dx = \int f(x)\ dx \pm \int g(x)\ dx$$

$$\int x^n\ dx = \frac{x^{n+1}}{n+1},\text{ where } n\neq -1$$

Section 1: Assignment Answer Key (36 marks)

1. $f(x) = 3x^2 - 6x^{\frac{1}{2}} + 1$

$$F(x) = \frac{3x^{2+1}}{2+1} - \frac{6x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 1x + C$$

$$= \frac{3x^3}{3} - \frac{6x^{\frac{3}{2}}}{\frac{3}{2}} + x + C$$

$$= x^3 - 4x^{\frac{3}{2}} + x + C \text{ (3 marks)}$$

2. $\int (2x-1)(x+4)\, dx = \int \left(2x^2 + 7x - 4\right) dx$

$$= \frac{2x^{2+1}}{2+1} + \frac{7x^{1+1}}{1+1} - 4x + C$$

$$= \frac{2x^3}{3} + \frac{7x^2}{2} - 4x + C \text{ (3 marks)}$$

3. Now $F(x) = \int (-x+1) + C$

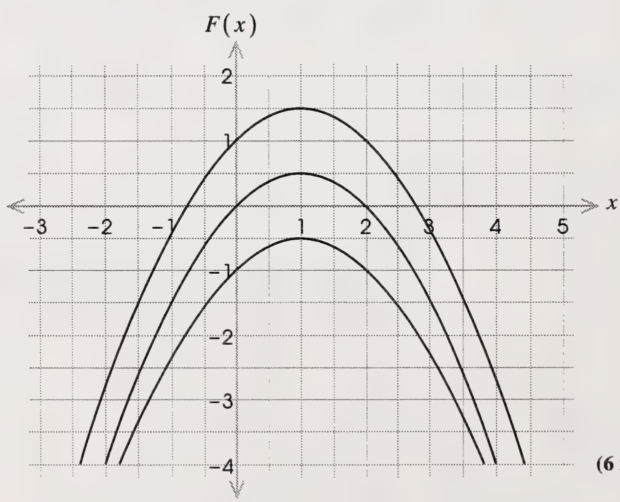
$$= -\frac{1}{2}x^2 + x + C$$

Answers may vary. Accept any three members and their graphs of the family described by the equation.

When $C = 0$, $F(x) = -\frac{1}{2}x^2 + x$

When $C = 1$, $F(x) = -\frac{1}{2}x^2 + x + 1$

When $C = -1$, $F(x) = -\frac{1}{2}x^2 + x - 1$



(6 marks)

4. $\dfrac{dy}{dx}=2x+1$

$$y=\int(2x+1)\,dx$$
$$=x^2+x+C$$

Replace x by 1 and y by -4.

$$-4=1^2+1+C$$
$$C=-6$$

$\therefore\ y=x^2+x-6$

To find the x-intercepts, replace y by 0.

$$x^2+x-6=0$$
$$(x-2)(x+3)=0$$
$$x-2=0 \quad\text{or}\quad x+3=0$$
$$x=2 \qquad\qquad x=-3$$

Therefore, the x-intercepts are 2 and -3. **(6 marks)**

5. $y''-4=0$

$$y''=4$$
$$y'=\int 4\,dx+C_1$$
$$y'=4x+C_1$$

$y'=0$ when $x=2$

$$0=4(2)+C_1$$
$$C_1=-8$$

$\therefore\ y'=4x-8$

$$y=\int(4x-8)\,dx+C_2$$
$$=2x^2-8x+C_2$$

$y=1$ when $x=2$

$$1=2(2)^2-8(-2)+C_2$$
$$1=8+16+C_2$$
$$C_2=-23$$

$\therefore\ y=2x^2-8x-23$ **(5 marks)**

6. Assume that the primitive is of the form $F(x)=\dfrac{a\left(3x^2-7\right)^{-2+1}}{-2+1}+C$

$$=-a\left(3x^2-7\right)^{-1}+C$$

Now, $\dfrac{d}{dx}F(x)=-a(-1)\left(3x^2-7\right)^{-1-1}(6x)+0$

$$=6ax\left(3x^2-7\right)^{-2}$$

Solve for a.

$$6ax\left(3x^2-7\right)^{-2}=x\left(3x^2-7\right)^{-2}$$

$$6a=1$$

$$a=\frac{1}{6}$$

$\therefore\ F(x)=-\frac{1}{6}\left(3x^2-7\right)^{-1}+C$ **(6 marks)**

7. a. $\int(4\cos 5x)\,dx=?$

Assume the primitive is of the form $F(x)=a\sin 5x+C$.

$$\frac{d}{dx}F(x)=a(\cos 5x)(5)+0$$

$$=5a\cos 5x$$

Equate the derivative to the integrand, and solve for a.

$$5a\cos 5x=4\cos 5x$$

$$5a=4$$

$$a=\frac{4}{5}$$

$\therefore\ \int(4\cos 5x)\,dx=\frac{4}{5}\sin 5x+C$ **(4 marks)**

b. $\int \left(\tan x \sec^2 x\right) dx = ?$

Since the derivative of $\tan x$ is $\sec^2 x$, apply the rule for integrating a power to the tangent function alone. Assume the primitive is of the form $F(x) = \frac{a}{2} \tan^2 x + C$.

Differentiate using the chain rule.

$$\begin{aligned} \frac{d}{dx} F(x) &= \frac{2a}{2}(\tan x)\left(\sec^2 x\right) + 0 \\ &= a \tan x \sec^2 x \end{aligned}$$

Equate the derivative to the integrand, and solve for a.

$$\begin{aligned} a \tan x \sec^2 x &= \tan x \sec^2 x \\ a &= 1 \end{aligned}$$

$\therefore \int \left(\tan x \sec^2 x\right) dx = \frac{1}{2} \tan^2 x + C$ **(3 marks)**

Section 2: Areas

Key Concepts

- inscribed and circumscribed rectangles
- limits of sums of inscribed and circumscribed rectangles
- the Fundamental Theorem of Calculus
- definite integrals
- area between a curve and the x-axis
- area between curves

The basic goals of this section are to ensure that students

- subdivide the area under a curve, over a given interval, into strips, and approximate the area of each strip by a rectangle
- approximate the area under a curve by finding the sum of inscribed or circumscribed rectangles
- establish the existence of upper and lower bounds for the area under a curve
- evaluate the exact area by finding the limit of the sums of either inscribed or circumscribed rectangles
- discuss the Fundamental Theorem of Calculus linking the area limit to the antiderivative of a function describing a curve
- calculate the definite integral for polynomial, rational, and trigonometric functions
- use the following properties to simplify more complicated definite integrals

$$\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx$$

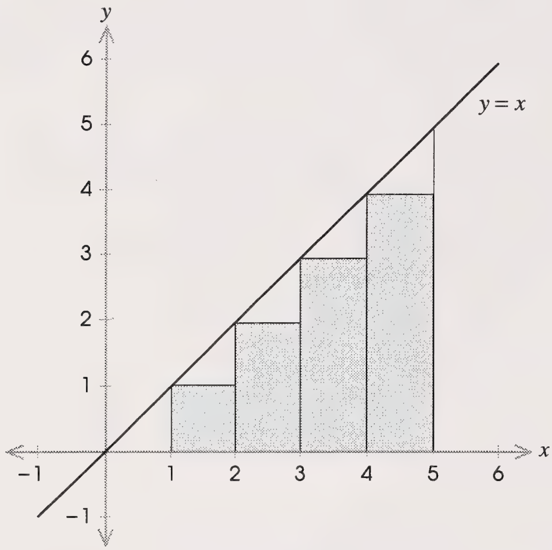
$$\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$$

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

- use the definite integral to find areas between a curve $y = f(x)$ and the x-axis, if the function has either a constant sign over the given interval or if the curve crosses the x-axis in that interval
- find areas between two curves either between two stated values or between their intersection points

Section 2: Assignment Answer Key (40 marks)

1. a.



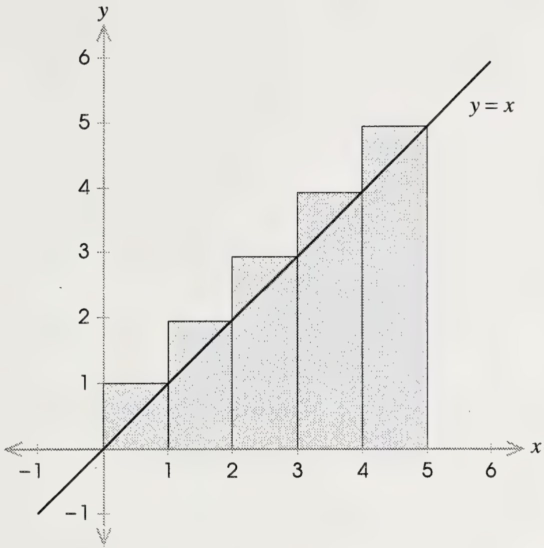
x	0	1	2	3	4
y	0	1	2	3	4

Rectangle Areas

Width	Length	Area
1	0	0
1	1	1
1	2	2
1	3	3
1	4	4
Total Area		10

The area approximated by the inscribed rectangles is less than the exact area. **(4 marks)**

b.



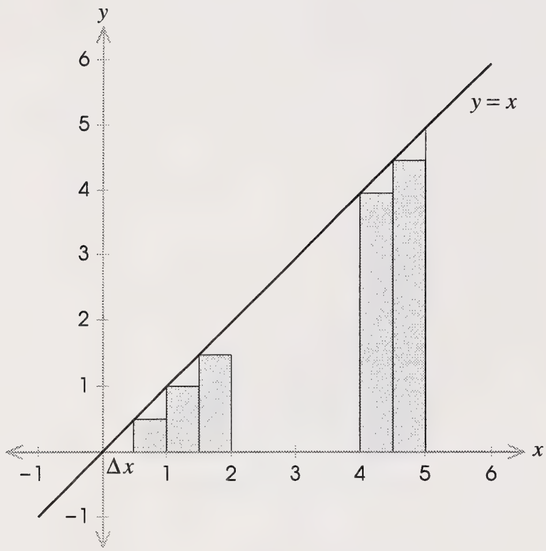
x	1	2	3	4	5
y	1	2	3	4	5

Rectangle Areas

Width	Length	Area
1	1	1
1	2	2
1	3	3
1	4	4
1	5	5
Total Area		15

The area approximated by the circumscribed rectangles is more than the exact area. **(4 marks)**

c.



The width of each strip is $\Delta x = \frac{5}{n}$.

	Width	Length	Area
Rectangle 1	$\frac{5}{n}$	0	0
Rectangle 2	$\frac{5}{n}$	$1\left(\frac{5}{n}\right)$	$1\left(\frac{5}{n}\right)^2$
Rectangle 3	$\frac{5}{n}$	$2\left(\frac{5}{n}\right)$	$2\left(\frac{5}{n}\right)^2$
Rectangle 4	$\frac{5}{n}$	$3\left(\frac{5}{n}\right)$	$3\left(\frac{5}{n}\right)^2$
⋮	⋮	⋮	⋮
Rectangle *n*	$\frac{5}{n}$	$(n-1)\left(\frac{5}{n}\right)$	$(n-1)\left(\frac{5}{n}\right)^2$

$$\text{Total Area} = 0 + 1\left(\frac{5}{n}\right)^2 + 2\left(\frac{5}{n}\right)^2 + 3\left(\frac{5}{n}\right)^2 + \ldots + (n-1)\left(\frac{5}{n}\right)^2$$

$$= \sum_{i=1}^{n}(i-1)\left(\frac{5}{n}\right)^2$$

As $n \to \infty$, the sum of the areas of the inscribed rectangles approaches the exact area. **(4 marks)**

d. $_0A_5 = \int_0^5 (x)\, dx$

$$= \left[\frac{x^2}{2}\right]_0^5$$

$$= \frac{5^2}{2} - \frac{0^2}{2}$$

$= 12.5$ **(2 marks)**

2. a. $\int_4^{12} \sqrt{2x+1}\, dx = \int_4^{12} (2x+1)^{\frac{1}{2}}\, dx$

Assume the primitive is of the form $F(x) = \frac{2a}{3}(2x+1)^{\frac{3}{2}} + C$.

$$\frac{d}{dx} F(x) = \frac{2a}{3}\left(\frac{3}{2}\right)(2x+1)^{\frac{1}{2}}(2) + 0$$

$$= 2a(2x+1)^{\frac{1}{2}}$$

Equate the derivative and the integrand.

$$2a(2x+1)^{\frac{1}{2}} = (2x+1)^{\frac{1}{2}}$$

$$2a = 1$$

$$a = \frac{1}{2}$$

$$\therefore\ F(x) = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C$$

$$\int_4^{12} \sqrt{2x+1}\, dx = \frac{1}{3}\left[(2x+1)^{\frac{3}{2}}\right]_4^{12}$$

$$= \frac{1}{3}\left[25^{\frac{3}{2}} - 9^{\frac{3}{2}}\right]$$

$$= \frac{1}{3}[125 - 27]$$

$= \frac{98}{3}$ **(3 marks)**

b. $\int_0^{\frac{\pi}{2}} 3\sin^2 x \cos x \, dx = ?$

Assume the primitive is of the form $F(x) = \frac{a\sin^3 x}{3}$.

$$\begin{aligned}\frac{d}{dx}F(x) &= \frac{3a \bullet \sin^2 x \bullet \cos x}{3} \\ &= a\sin^2 x \cos x\end{aligned}$$

Equate the derivative to the integrand.

$$\begin{aligned}a\sin^2 x \cos x &= 3\sin^2 x \cos x \\ a &= 3\end{aligned}$$

$\therefore F(x) = \sin^3 x$

$$\begin{aligned}\int_0^{\frac{\pi}{2}} 3\sin^2 x \cos x \, dx &= \left[\sin^3 x\right]_0^{\frac{\pi}{2}} \\ &= \sin^3 \frac{\pi}{2} - \sin^3 0 \\ &= 1^3 - 0^3 \\ &= 1 \quad \textbf{(3 marks)}\end{aligned}$$

3. Since the function $y = \sqrt{3x+1}$ is non-negative, ${}_0A_1 = \int_0^1 \sqrt{3x+1}\, dx$.

Assume the primitive is of the form $F(x) = \frac{2a(3x+1)^{\frac{3}{2}}}{3}$.

$$\begin{aligned}\frac{d}{dx}F(x) &= \frac{2a}{3}\left(\frac{3}{2}\right)(3x+1)^{\frac{1}{2}}(3) \\ &= 3a(3x+1)^{\frac{1}{2}}\end{aligned}$$

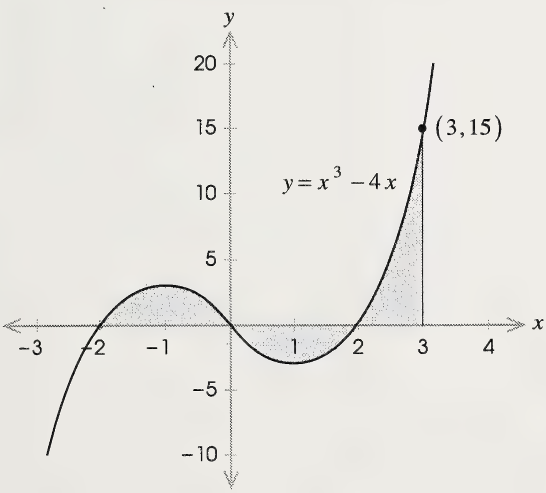
Equate $3a(3x+1)^{\frac{1}{2}}$ to the integrand.

$$\begin{aligned}3a(3x+1)^{\frac{1}{2}} &= (3x+1)^{\frac{1}{2}} \\ 3a &= 1 \\ a &= \frac{1}{3}\end{aligned}$$

$\therefore F(x) = \frac{2}{9}(3x+1)^{\frac{3}{2}}$

$$ {}_0A_1 = \frac{2}{9}\left[(3x+1)^{\frac{3}{2}} \right]_0^1 $$
$$ = \frac{2}{9}\left[4^{\frac{3}{2}} - 1^{\frac{3}{2}} \right] $$
$$ = \frac{2}{9}[8-1] $$
$$ = \frac{14}{9} $$ **(4 marks)**

4.



Find the x-intercepts of $y = x^3 - 4x$.

Let $y = 0$.

$$ x^3 - 4x = 0 $$
$$ x(x^2 - 4) = 0 $$
$$ x(x+2)(x-2) = 0 $$
$$ x = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 2 = 0 $$
$$ x = -2 \qquad\qquad x = 2 $$

The area from -2 to 0 is positive; the area from $x = 0$ to $x = 2$ is negative. The area from $x = 2$ to $x = 3$ is positive.

$$\text{Total Area} = {}_{-2}A_0 + {}_0A_2 + {}_2A_3$$

$$= \int_{-2}^{0} \left(x^3 - 4x\right) dx + \left|\int_{0}^{2} \left(x^3 - 4x\right) dx\right| + \int_{2}^{3} \left(x^3 - 4x\right) dx$$

$$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} + \left|\frac{x^4}{4} - 2x^2\right|_{0}^{2} + \left[\frac{x^4}{4} - 2x^2\right]_{2}^{3}$$

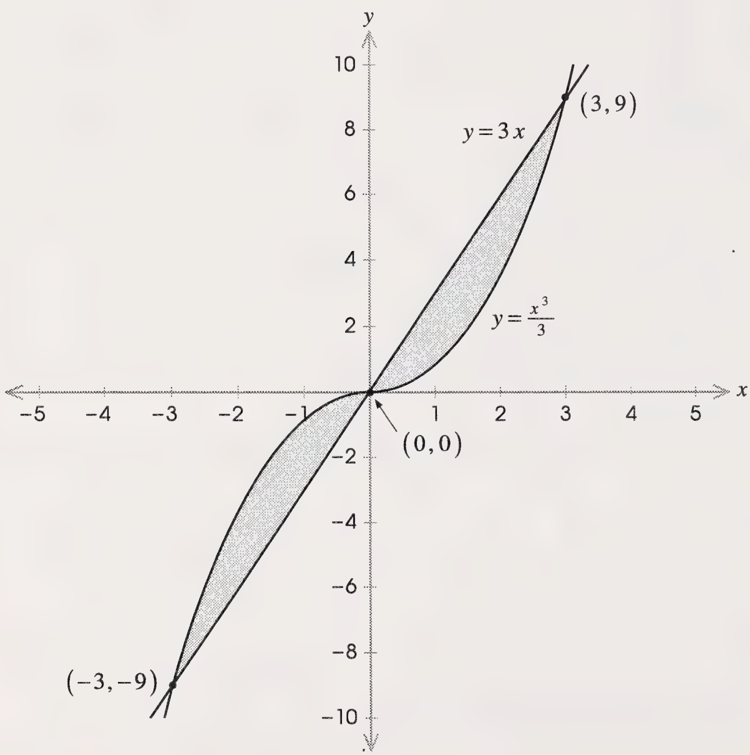
$$= \left[\left(\frac{0^4}{4} - 2(0)^2\right) - \left(\frac{(-2)^4}{4} - 2(-2)^2\right)\right] + \left|\frac{2^4}{4} - 2(2)^2\right| + \left[\left(\frac{3^4}{4} - 2(3)^2\right) - \left(\frac{2^4}{4} - 2(2)^2\right)\right]$$

$$= \left[0 - (4 - 8)\right] + \left|4 - 8\right| + \left[\left(\frac{81}{4} - 18\right) - (4 - 8)\right]$$

$$= 4 + 4 + \frac{81}{4} - 14$$

$= 14.25$ **(6 marks)**

5.



Determine the points of intersection of $y = \frac{x^3}{3}$ and $y = 3x$.

Let $f(x) = \frac{x^3}{3}$ and $g(x) = 3x$.

$$\frac{x^3}{3} = 3x$$

$$x^3 = 9x$$

$$x^3 - 9x = 0$$

$$x(x-3)(x+3) = 0$$

$$x = 0 \quad \text{or} \quad x - 3 = 0 \quad \text{or} \quad x + 3 = 0$$

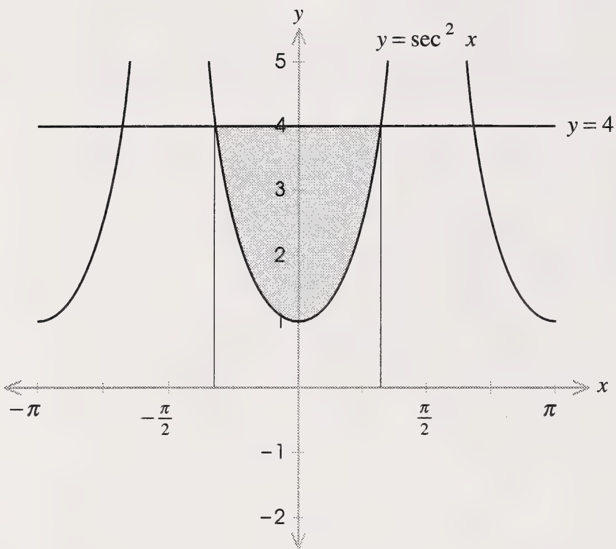
$$x = 3 \qquad\qquad x = -3$$

When $x = 0$, $y = 0$.
When $x = -3$, $y = -9$.
When $x = 3$, $y = 9$.

$$
\begin{aligned}
{}_{-3}A_3 &= {}_{-3}A_0 + {}_0A_3 \\
&= \int_{-3}^{0} [f(x) - g(x)]\, dx + \int_{0}^{3} [g(x) - f(x)]\, dx \\
&= \int_{-3}^{0} \left(\frac{x^3}{3} - 3x \right) dx + \int_{0}^{3} \left(3x - \frac{x^3}{3} \right) dx \\
&= \left[\frac{x^4}{12} - \frac{3x^2}{2} \right]_{-3}^{0} + \left[\frac{3x^2}{2} - \frac{x^4}{12} \right]_{0}^{3} \\
&= \left[\left(\frac{0^4}{12} - \frac{3(0)^2}{2} \right) - \left(\frac{(-3)^4}{12} - \frac{3(-3)^2}{2} \right) \right] + \left[\left(\frac{3(3)^2}{2} - \frac{3^4}{12} \right) - \left(\frac{3(0)^2}{2} - \frac{0^4}{12} \right) \right] \\
&= \left[0 - \left(\frac{81}{12} - \frac{27}{2} \right) \right] + \left[\left(\frac{27}{2} - \frac{81}{12} \right) - 0 \right] \\
&= \left[0 - \left(\frac{27}{4} - \frac{54}{4} \right) \right] + \left[\left(\frac{54}{4} - \frac{27}{4} \right) \right] \\
&= \frac{27}{4} + \frac{27}{4} \\
&= \frac{27}{2}
\end{aligned}
$$

(6 marks)

6.



$$\sec^2 x = 4$$
$$\frac{1}{\cos^2 x} = 4$$
$$\cos^2 x = \frac{1}{4}$$
$$\cos x = \pm\frac{1}{2}$$

Since $\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, the curves intersect at the limits of integration. There is a single area enclosed, as shown in the diagram.

$$
\begin{aligned}
{}_{-\frac{\pi}{3}}A_{\frac{\pi}{3}} &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(4 - \sec^2 x\right) dx \\
&= [4x - \tan x]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\
&= \left[\frac{4\pi}{3} - \tan\frac{\pi}{3}\right] - \left[4\left(-\frac{\pi}{3}\right) - \tan\left(-\frac{\pi}{3}\right)\right] \\
&= \left[\frac{4\pi}{3} - \sqrt{3}\right] - \left[-\frac{4\pi}{3} + \sqrt{3}\right] \\
&= \frac{8\pi}{3} - 2\sqrt{3} \quad \textbf{(4 marks)}
\end{aligned}
$$

Section 3: Applications of Integration

Key Concepts

- distance, position, and displacement
- velocity
- acceleration
- harmonic motion
- mean or average value of a function

The basic goals of this section are to ensure that students

- use the antiderivatives of acceleration and velocity functions to obtain velocity and displacement functions
- solve problems associated with position, velocity, and acceleration that were posed in the forms $v = f(t)$ or $a = f(t)$
- derive the kinematic equations

$$v = at + v_0$$
$$v^2 = v_0^2 + 2ad$$
$$d = \frac{1}{2}at^2 + v_0 t + d_0$$

- determine the equations of velocity and acceleration in simple harmonic motion starting from the displacement equation $x = A\cos(bt + C)$
- work backwards from the second-order differential equation modelling harmonic motion to obtain velocity and displacement
- calculate the mean value of a function over an interval

Section 3: Assignment Answer Key (14 marks)

1. $v = 3t - 24t^2$

$$s = \int\left(3t - 24t^2\right)dt + C$$
$$= \frac{3t^2}{2} - 8t^3 + C$$

$s = 2$ when $t = 2$

$$2 = \frac{3(2)^2}{2} - 8(2)^3 + C$$
$$2 = 6 - 64 + C$$
$$C = 60$$

$\therefore\ s = \frac{3t^2}{2} - 8t^3 + 60$ **(3 marks)**

2. Find the velocity function.

$$\begin{aligned} v &= \int (-10)\ dt + C \\ &= -10t + C \end{aligned}$$

$v = 0$ at $t = 0$

$$\begin{aligned} 0 &= -10(0) + C \\ C &= 0 \end{aligned}$$

$\therefore\ v = -10t$

The rock does not change direction. The distance may be calculated using a definite integral.

$$\begin{aligned} \text{Total Distance} &= \left| \int_2^4 (-10t)\ dt \right| \\ &= \left| -5t^2 \Big|_2^4 \right| \\ &= \left| -5(4)^2 - \left(-5(2)^2\right) \right| \\ &= 60 \end{aligned}$$

The rock travelled 60 m. **(3 marks)**

3. $a = -0.01(1024\pi)^2 \cos 1024\ \pi t$

$$\begin{aligned} v &= \int a\ dt + C_1 \\ &= \int \left[-0.01(1024\pi)^2 \cos 1024\pi t \right] dt + C_1 \\ &= \frac{-0.01\left[1024\pi t^2 (\sin 1024\pi t)\right]}{1024\pi} + C_1 \\ &= -0.01(1024\pi)\ \sin 1024\pi t + C_1 \end{aligned}$$

$v = 0$ at $t = 0$

$$\begin{aligned} 0 &= -0.01(1024\pi)\ \sin 1024\pi(0) + C_1 \\ C_1 &= 0 \end{aligned}$$

$\therefore\ v = -0.01(1024\pi)\ \sin 1024\pi t$ **(2 marks)**

$$s = \int v \, dt$$

$$s = \int \left[-0.01(1024\pi) \sin 1024\pi t \right] dt + C_2$$

$$= \frac{0.01(1024\pi)(-\cos 1024\pi t)}{1024\pi} + C_2$$

$$= 0.01 \cos 1024\pi t + C_2$$

$s = 0.01$ at $t = 0$

$$0.01 = 0.01 \cos 1024\pi(0) + C_2$$

$$0.01 = 0.01 + C_2$$

$$C_2 = 0$$

$\therefore \ s = 0.01 \cos 1024\pi t$ **(2 marks)**

4. $\left(y_{\text{ave}}\right)_x = \dfrac{1}{b-a} \displaystyle\int_a^b f(x) \, dx$

In this question, $f(x) = 3x^2 - x$, $a = 1$, and $b = 2$.

$$\therefore \ \left(y_{\text{ave}}\right)_x = \frac{1}{2-1} \int_1^2 \left(3x^2 - x\right) dx$$

$$= \left[x^3 - \frac{x^2}{2} \right]_1^2$$

$$= \left[2^3 - \frac{2^2}{2} \right] - \left[1^3 - \frac{1^2}{2} \right]$$

$$= [8-2] - \left[1 - \frac{1}{2} \right]$$

$$= 6 - \frac{1}{2}$$

$= 5.5$ **(4 marks)**

Final Module Assignment (10 marks)

1. Both $y = x - 3$ and $y = x + 1$ have the same slope, since the derivative of the constant term in either equation is 0.

$$\begin{aligned}\frac{d}{dx}(x-3) &= 1+0 \\ &= 1\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(x+2) &= 1+0 \\ &= 1\end{aligned}$$

Substitute the value of the derivative into the differential equation.

$$\begin{aligned}\therefore\ \frac{dy}{dx} - 1 &= 1 - 1 \\ &= 0 \quad \textbf{(3 marks)}\end{aligned}$$

2. Veronica was right because the function crosses the x-axis in the interval $[0, 3]$ at $x = 1$. Therefore, the definite integral represents the net sum of positive and negative areas. The total area must be calculated by adding the absolute values of all positive and negative areas. **(3 marks)**

3. The object changes direction when $v = 0$.

$\therefore\ t - 3 = 0$ or $t = 3$

$$\begin{aligned}\text{Total Distance} &= \left|\int_0^3 (t-3)\ dt\right| + \int_3^4 (t-3)\ dt \\ &= \left|\frac{t^2}{2} - 3t\right|_0^3 + \left[\frac{t^2}{2} - 3t\right]_3^4 \\ &= \left|\frac{9}{2} - 9\right| + \left[\left(\frac{16}{2} - 12\right) - \left(\frac{9}{2} - 9\right)\right] \\ &= \frac{9}{2} + \frac{7}{2} - 3 \\ &= 5 \quad \textbf{(4 marks)}\end{aligned}$$

Module 8: Exponential and Logarithmic Functions

Overview

In this module the student investigates exponential and logarithmic functions—both from a theoretical point of view. In Section 1, the natural logarithm $(\ln x)$ is defined as an area under the graph of $y = \frac{1}{t}$, from $t = 1$ to $t = x$. This approach is taken to complete the discussion of integrals of powers $\left(x^n\right)$, which began in Module 7.

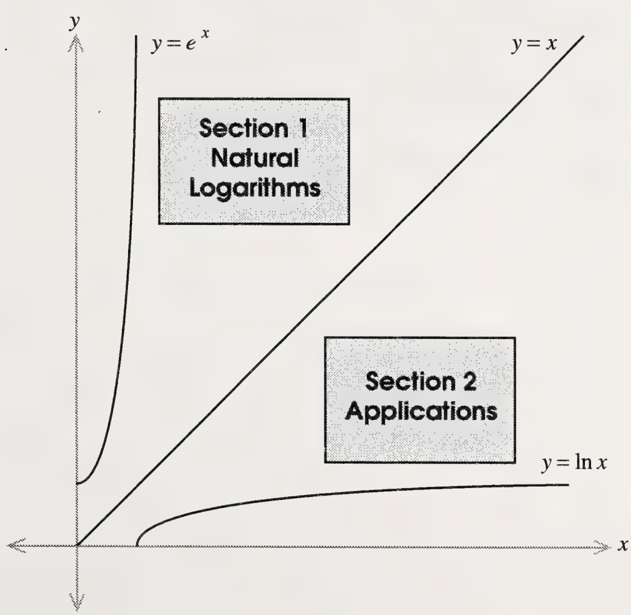
The properties of natural logarithms are developed from this perspective. The exponential function is defined as the natural logarithm's inverse function. Differentiation and integration techniques, fundamental to this discussion, occur in each activity. Methods for approximating the base of the natural logarithm, the discussion of limits, and the relationships among logarithms and exponential functions of different bases round out Section 1.

Many of the applications from Mathematics 30 are reintroduced in Section 2. Compound interest, exponential growth and decay, light absorption, and heating and cooling are a few of the everyday situations analysed.

After completing this module, the student is expected to represent both exponential and logarithmic functions symbolically and graphically, and should be able to differentiate and integrate various other functions in which they appear. The student is also expected to model real-world situations and solve problems using these functions.

It may be helpful for the student to review the module on logarithmic and exponential functions from Mathematics 30.

Module 8
Exponential and Logarithmic Functions



Evaluation

The evaluation of this module will be based on three assignments:

Section 1 Assignment	**67 marks**
Section 2 Assignment	**15 marks**
Final Module Assignment	**18 marks**
TOTAL	**100 marks**

Section 1: Natural Logarithms

Key Concepts

- natural logarithms
- inverse functions
- base-e exponential function
- continuous compound interest
- limits of exponential and logarithmic functions
- estimation of e and e^x
- change of base

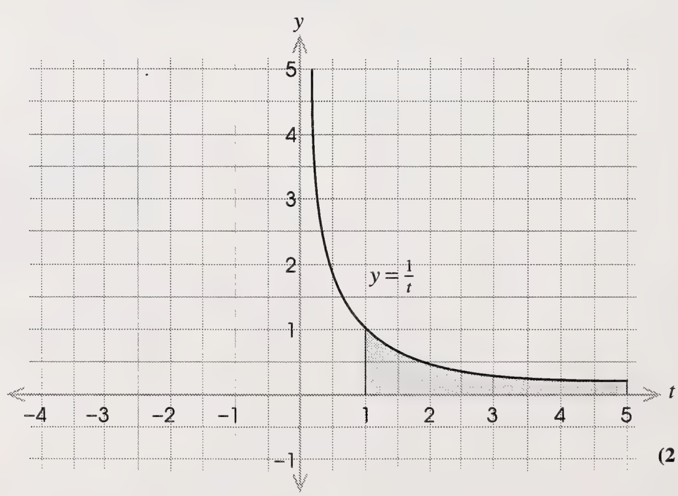
The basic goals of this section are to ensure that students

- define exponential and logarithmic functions
- recognize that exponential and logarithmic functions are inverses
- differentiate logarithmic and exponential functions
- analyse the graphs of $y = \ln x$ and $y = e^x$, and approximate the slopes of those graphs for various values of x
- use limit theorems to evaluate the limits of simple exponential and logarithmic functions
- explain that e may be defined as a limit, and estimate the values of the limits of e and e^x
- find the derivatives of logarithmic functions having bases other than e
- find the derivatives and antiderivatives of exponential functions having bases other than e
- evaluate maxima and minima of given functions involving exponential and logarithmic functions
- find areas bounded by exponential, logarithmic, or reciprocal functions

Section 1: Assignment Answer Key (67 marks)

1. **a.** $\ln 5 = \int_1^5 \frac{dt}{t}$ **(1 mark)**

b.



(2 marks)

2. **a.** $$\frac{dy}{dx}=\frac{1}{(3x-2)^2}\bullet\frac{d}{dx}(3x-2)^2$$
$$=\frac{2(3x-2)(3)}{(3x-2)^2}$$
$$=\frac{6}{3x-2}$$ **(3 marks)**

b. Apply the product rule.

$$\frac{dy}{dx}=x\bullet\frac{d}{dx}(\ln x)^2+(\ln x)^2\bullet\frac{d}{dx}(x)$$
$$=x(2)(\ln x)\left(\frac{1}{x}\right)+[\ln x]^2\,(1)$$
$$=2\ln x+(\ln x)^2$$ **(3 marks)**

3. **a.** The antiderivative of $f(x)=\frac{2}{x}$ must be written as $F(x)=2\ln|x|$, since natural logarithms are not defined for negative numbers. **(1 mark)**

b. $$F(x)=2\ln|x|$$
$$=\ln|x|^2$$
$$=\ln x^2$$ **(2 marks)**

4. **a.** $$\int\frac{6x^2}{x^3-1}\,dx$$

Assume the primitive is of the form $F(x)=a\ln\left|x^3-1\right|+C$.

$$\frac{d}{dx}F(x)=\frac{a}{x^3-1}\left(3x^2\right)$$
$$=\frac{3ax^2}{x^3-1}$$

Solve for a.

$$\frac{3ax^2}{x^3-1}=\frac{6x^2}{x^3-1}$$
$$3a=6$$
$$a=2$$

$$\therefore\ \int\frac{6x^2}{x^3-1}\,dx=2\ln\left|x^3-1\right|+C$$ **(4 marks)**

b. $\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$

$= \ln|\sin x| + C$ **(2 marks)**

5. $y = \ln(3-2x)$ is defined if $3-2x>0$.

$$\begin{aligned} 3-2x &> 0 \\ 3 &> 2x \\ 2x &< 3 \\ x &< 1.5 \end{aligned}$$

The domain is $(1.5, \infty)$. **(3 marks)**

6. To determine the equation of the tangent to $y = \ln x$ at $x = e$, find the slope of $y = \ln x$ at $x = e$ first.

$$\frac{dy}{dx} = \frac{1}{x}$$

At $x = e$, the slope of the tangent is $\frac{1}{e}$.

Next, find the point of contact.

When $x = e$, $y = \ln e = 1$. The point of tangency is $(e, 1)$.

Use $y - y_1 = m(x - x_1)$ to find the equation.

$m = \frac{1}{e}$ or e^{-1}, $x_1 = e$, and $y_1 = 1$

$$\begin{aligned} y-1 &= e^{-1}(x-e) \\ y-1 &= e^{-1}x - 1 \\ \therefore y &= e^{-1}x \end{aligned}$$ **(5 marks)**

7. $$\begin{aligned} \ln 0.12 &= \ln\left(\frac{12}{100}\right) \\ &= \ln\left(\frac{3}{25}\right) \\ &= \ln 3 - \ln 25 \\ &= \ln 3 - \ln 5^2 \\ &= \ln 3 - 2\ln 5 \\ &= b - 2a \end{aligned}$$ **(3 marks)**

8. $\frac{\ln a}{2} - \ln b + 3\ln c = \ln \frac{\sqrt{a}\left(c^3\right)}{b}$ **(2 marks)**

9. Original function: $y = 3\ln(x-1)$

Inverse: $x = 3\ln(y-1)$

$$x = \ln(y-1)^3$$

$$(y-1)^3 = e^x$$

$$y-1 = e^{\frac{x}{3}}$$

$$y = e^{\frac{x}{3}} + 1$$ **(3 marks)**

10. **a.** $\ln e^{0.5} = 0.5$ **(1 mark)**

b. $3e^{\ln 4} = 12$ **(1 mark)**

11. To determine the equation of the normal to $y = e^x$ at $x = 0$, first find the slope of at $x = 0$.

$$\frac{dy}{dx} = e^x$$

At $x = 0$, $\frac{dy}{dx} = e^0 = 1$.

Therefore, the slope of the normal is -1.

The normal intersects the graph at $x = 0$. When $x = 0$, $y = e^0 = 1$.

Use $y - y_1 = m(x - x_1)$, where $x_1 = 0$, $y_1 = 1$, and $m = -1$.

$$y - 1 = -1(x - 0)$$

$$y = -x + 1$$ **(5 marks)**

12. The tangent to $y = e^x$ intersects the x-axis at $(x-1, 0)$. Therefore, the x-intercept is $(45-1)$ or 44. **(2 marks)**

13. Apply the product rule.

$$\frac{dy}{dx} = x\frac{d}{dx}\left(e^{\sin x}\right) + e^{\sin x}\frac{d}{dx}(x)$$

$$= x(\cos x)e^{\sin x} + e^{\sin x}$$ **(3 marks)**

14. $_0A_{\frac{\pi}{2}} = \int_0^{\frac{\pi}{2}} (\cos x) e^{\sin x}\, dx$

$$= \left[e^{\sin x} \right]_0^{\frac{\pi}{2}}$$

$$= e^{\sin\left(\frac{\pi}{2}\right)} - e^{\sin 0}$$

$$= e^1 - e^0$$

$$= e - 1$$ **(3 marks)**

15. Assume the primitive is of the form $F(x) = \frac{ae^{7x}}{7} + C$.

$$\frac{d}{dx} F(x) = \frac{7ae^{7x}}{7}$$

Solve for a.

$$ae^{7x} = 3e^{7x}$$

$$a = 3$$

$\therefore \int 3e^{7x}\, dx = 3e^{7x} + C$ **(3 marks)**

16. **a.** $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x = e^2$

$\doteq 7.389$ **(1 mark)**

Note: Accept either answer.

b. $\lim_{h \to 0} (1 + 3h)^{\frac{1}{h}} = e^3$

$\doteq 20.086$ **(1 mark)**

Note: Accept either answer.

c. $\lim_{x \to +\infty} 3^{-x} = 0$ **(1 mark)**

d. $\lim_{x \to 2^+} \ln(x-2) = -\infty$ **(1 mark)**

17. $A = Pe^{rt}$, where $A = 3P$ and $t = 5$

$$3P = Pe^{5r}$$

$$e^{5r} = 3$$

$$\ln e^{5r} = \ln 3$$

$$5r = \ln 3$$

$$r = \frac{\ln 3}{5}$$

$$\doteq 0.22$$

An interest rate of about 22% will triple the size of your account. **(3 marks)**

18. $\log_7 13 = \dfrac{\ln 13}{\ln 7}$ **(1 mark)**

19. $13^n = e^{(\ln 13)n}$ **(1 mark)**

20. **a.** $y = x2^x$

$$\frac{dy}{dx} = x\frac{d}{dx}\left(2^x\right) + 2^x\frac{d}{dx}(x)$$
$$= x(\ln 2)2^x + 2^x \quad \textbf{(2 marks)}$$

b. $y = \log(2x)$

$$\frac{dy}{dx} = \frac{1}{2x\ln 10}(2)$$
$$= \frac{1}{x\ln 10} \quad \textbf{(2 marks)}$$

21. $\int 10^x \, dx = \dfrac{10^x}{\ln 10} + C$ **(2 marks)**

Section 2: Applications

Key Concepts

- natural or exponential growth
- natural or exponential decay

The basic goals of this section are to ensure that students

- relate natural growth and decay to the differential equations $y' = ky$ or $y' = k\left(y - y_0\right)$
- solve natural growth and decay problems starting from the differential equations $y' = ky$ or $y' = k\left(y - y_0\right)$
- fit exponential models to observed data

Section 2: Assignment Answer Key (15 marks)

1. Begin by finding the constant k.

$y = y_0 e^{kt}$, where $y_0 = 40$ million, and $y = 160$ million at $t = 20$ d

$$\therefore\ 160 = 40e^{k(20)}$$
$$e^{20k} = 4$$
$$20k = \ln 4$$
$$k = \frac{\ln 4}{20}$$

$$\therefore\ y = 40e^{\frac{(\ln 4)t}{20}}$$

After $t = 50$ d, $y = 40e^{\frac{(\ln 4)50}{20}}$
$= 1280$

The population of grasshoppers will be 1280 million. **(5 marks)**

2. Begin by finding the constant *k*.

$y = y_0 e^{kt}$, where $y_0 = 30$ g, and $y = 20$ g at $t = 10$ min

$$\begin{aligned}
\therefore\ 20 &= 30e^{k(10)}\\
e^{10k} &= \frac{2}{3}\\
10k &= \ln\left(\frac{2}{3}\right)\\
k &= \frac{\ln\left(\frac{2}{3}\right)}{10}
\end{aligned}$$

$$\therefore\ y = 30e^{\frac{\ln\left(\frac{2}{3}\right)}{10}t}$$

To find the half-life, solve for *t* when $y = 15$ g.

$$\begin{aligned}
15 &= 30e^{\frac{\ln\left(\frac{2}{3}\right)}{10}t}\\
0.5 &= e^{\frac{\ln\left(\frac{2}{3}\right)}{10}t}\\
\frac{t\left[\ln\left(\frac{2}{3}\right)\right]}{10} &= \ln 0.5\\
t &= \frac{10\ln 0.5}{\ln\left(\frac{2}{3}\right)}\\
&\doteq 17.1
\end{aligned}$$

The half-life is approximately 17.1 min. **(5 marks)**

3. $y = 10\,000e^{-0.01t}$

The derivative yields the rate at which the population is changing.

$$\begin{aligned}
\frac{dy}{dt} &= 10\,000(-0.01)e^{-0.01t}\\
&= -100e^{-0.01t}
\end{aligned}$$

Find the rate of change at $t = 3$ min.

$$\frac{dy}{dt} = -100\,e^{-0.03}$$
$$= -97$$

The population is decreasing at the rate of 97 bacteria/minute.

This is an example of natural decay, since the population is declining. **(5 marks)**

Final Module Assignment (18 marks)

1. The domain is the set of reals. **(1 mark)**

2. There are no x-intercepts.

When $x = 0,\ y = 2e^{0} = 2$. Therefore, the graph intersects the y-axis at $(0, 2)$. **(2 marks)**

3. Since the function is not changed when x is replaced by $-x$, the graph is symmetric with respect to the y-axis. **(1 mark)**

4. As $x \to \infty,\ y \to 0$. Therefore, the x-axis is an asymptote to the graph. **(1 mark)**

5. Intervals of increase and decrease $y = 2e^{-0.5x^2}$.

$$\frac{dy}{dx} = 2e^{-0.5x^2}(-0.5)(2x)$$
$$\frac{dy}{dx} = -2xe^{-0.5x^2}$$

The sign of the first derivative depends on the sign of x.

When $x > 0,\ \frac{dy}{dx} < 0$, and the graph falls to the right.

When $x < 0,\ \frac{dy}{dx} > 0$, and the graph rises to the right. **(4 marks)**

6. The graph has a maximum point at $(0, 2)$. **(1 mark)**

7. $\frac{dy}{dx} = -2xe^{-0.5x^2}$

$$\frac{d^2y}{dx^2} = -2x\frac{d}{dx}\left(e^{-0.5x^2}\right) + e^{-0.5x^2}\frac{d}{dx}(-2x)$$
$$= -2x\left(-xe^{-0.5x^2}\right) - 2e^{-0.5x^2}$$
$$= \left(2x^2 - 2\right)e^{-0.5x^2}$$
$$= 2\left(x^2 - 1\right)e^{-0.5x^2}$$

The second derivative is positive when $x^2 - 1 > 0$.

$$x^2 > 1$$
$$x > 1 \text{ or } x < 1$$

Therefore, the curve is concave upward on the interval $(-\infty, 1) \cup (1, \infty)$.

The second derivative is negative when $x^2 - 1 < 0$ or $-1 < x < 1$.

Therefore, the curve is concave downward on the interval $(-1, 1)$. **(4 marks)**

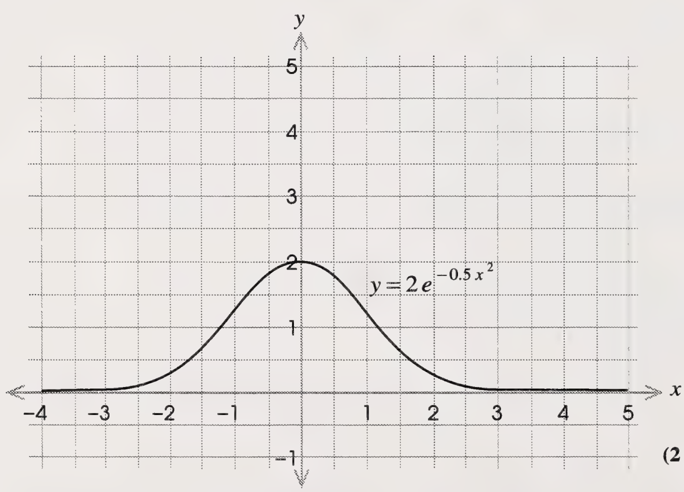
8. Points of inflection occur when $y'' = 0$.

$$x^2 - 1 = 0$$
$$\therefore\ x = \pm 1$$

When $x = \pm 1$, $y = 2e^{-0.5}$.

The points are $\left(1, 2e^{-0.5}\right)$ and $\left(-1, 2e^{-0.5}\right)$. **(2 marks)**

9. The graph is as follows:



(2 marks)

Final Test

Included here is the answer key to the final test and the student's copy of the final test. The student's copy of the final test is designed for photocopying and faxing.

Note

The answer key and student's copy of this final test should be kept secure by the teacher. Students should not have access to this test until it is assigned in a supervised situation. The answers should be stored securely and retained by the teacher at all times.

MATHEMATICS 31

FINAL TEST ANSWER KEY

Part A: Multiple Choice (80 marks)

Each multiple choice question is worth 2 marks.

1.	A	**9.**	B	**17.**	C	**25.**	C	**33.**	C
2.	D	**10.**	A	**18.**	A	**26.**	A	**34.**	B
3.	B	**11.**	D	**19.**	B	**27.**	B	**35.**	B
4.	C	**12.**	B	**20.**	B	**28.**	C	**36.**	B
5.	D	**13.**	C	**21.**	A	**29.**	B	**37.**	C
6.	B	**14.**	C	**22.**	D	**30.**	D	**38.**	D
7.	A	**15.**	D	**23.**	D	**31.**	A	**39.**	D
8.	D	**16.**	D	**24.**	A	**32.**	B	**40.**	C

Part B: Written Response (20 marks)

1. **a.** $x^2+y^2=5$ (1)

$y=\sqrt{2x-3}$ (2)

Substitute $\sqrt{2x-3}$ for y in (1).

$$\begin{aligned} x^2+\left(\sqrt{2x-3}\right)^2 &= 5 \\ x^2+2x-3 &= 5 \\ x^2+2x-8 &= 0 \\ (x+4)(x-2) &= 0 \\ \therefore\ x &= -4 \text{ or } 2 \end{aligned}$$

Since $y=\sqrt{2x-3}$ is undefined when $x=-4$, the x-coordinate of the point of intersection is 2. To find the y-value, substitute in (2).

When $x=2$, $y=\sqrt{2(2)-3}$

$=\sqrt{1}$

$=1$

The point of intersection is $(2,1)$. **(3 marks)**

b. To find the slope m_1, differentiate $x^2 + y^2 = 5$ at $(2,1)$ implicitly.

$$2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

At $(2,1)$, $m_1 = -\frac{2}{1}$
$= -2$

To find the slope m_2, differentiate $y = \sqrt{2x-3}$ at $(2,1)$.

$$y = (2x-3)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2}(2x-3)^{-\frac{1}{2}}(2)$$
$$= (2x-3)^{-\frac{1}{2}}$$

At $(2,1)$, $m_2 = [2(2)-3]^{-\frac{1}{2}}$
$= 1^{-\frac{1}{2}}$
$= 1$

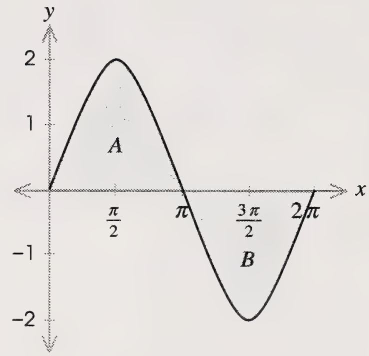
Therefore, the slopes of m_1 and m_2 are -2 and 1, respectively. **(4 marks)**

c. $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

$$= \frac{1-(-2)}{1-(-2)(1)}$$
$$= 1$$

$\therefore\ \theta = 45°$ **(1 mark)**

2. Answers will vary. The definite integral from 0 to 2π is 0, because areas A and B in the diagram are equal in magnitude but opposite in sign. The definite integral from 0 to π is positive, because area A lies above the x-axis. **(3 marks)**



3. $y = \sqrt{3-5x}$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3-5(x+h)}-\sqrt{3-5x}}{h}$$

$$= \lim_{h\to 0} \frac{\left[\sqrt{3-5(x+h)}-\sqrt{3-5x}\right]}{h} \bullet \frac{\left[\sqrt{3-5(x+h)}+\sqrt{3-5x}\right]}{\left[\sqrt{3-5(x+h)}+\sqrt{3-5x}\right]}$$

$$= \lim_{h\to 0} \frac{\left[3-5(x+h)\right]-(3-5x)}{h\left[\sqrt{3-5(x+h)}+\sqrt{3-5x}\right]}$$

$$= \lim_{h\to 0} \frac{3-5x-5h-3+5x}{h\left[\sqrt{3-5(x+h)}+\sqrt{3-5x}\right]}$$

$$= \lim_{h\to 0} \frac{-5h}{h\left[\sqrt{3-5(x+h)}+\sqrt{3-5x}\right]}$$

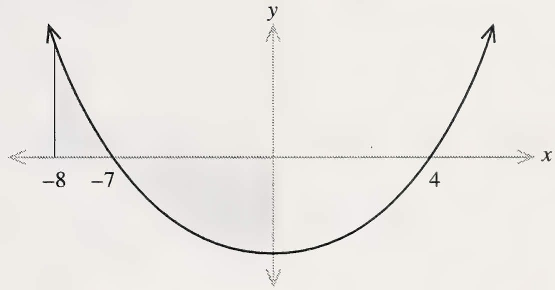
$$= \lim_{h\to 0} \frac{-5}{\sqrt{3-5(x+h)}+\sqrt{3-5x}}$$

$$= \frac{-5}{\sqrt{3-5x}+\sqrt{3-5x}}$$

$$= \frac{-5}{2\sqrt{3-5x}}$$ **(4 marks)**

4. Determine the x-intercepts.

$$x^2 + 3x - 28 = 0$$
$$(x+7)(x-4) = 0$$
$$x+7=0 \quad \text{or} \quad x-4=0$$
$$x=-7 \qquad\qquad x=4$$



$$\begin{aligned}{}_{-8}A_{-7} &= \int_{-8}^{-7}\left(x^2+3x-28\right)dx\\ &= \left[\frac{x^3}{3}+\frac{3}{2}x^2-28x\right]_{-8}^{-7}\\ &= \left(\frac{-343}{3}+\frac{147}{2}+196\right)-\left(\frac{-512}{3}+\frac{192}{2}+224\right)\\ &= \frac{-686+441+1176+1024-576-1344}{6}\\ &= \frac{35}{6}\end{aligned}$$

$$\begin{aligned}{}_{-7}A_{0} &= \int_{-7}^{0}\left(x^2+3x-28\right)dx\\ &= \left[\frac{x^3}{3}+\frac{3}{2}x^2-28x\right]_{-7}^{0}\\ &= 0-\left(\frac{-343}{3}+\frac{147}{2}+196\right)\\ &= -\left(\frac{-686+441+1176}{6}\right)\\ &= -\left(\frac{931}{6}\right)\end{aligned}$$

$$\begin{aligned}\text{Total Area} &= \left|\frac{35}{6}\right|+\left|\frac{-931}{6}\right|\\ &= \frac{966}{6}\\ &= 161 \quad \textbf{(5 marks)}\end{aligned}$$

MATHEMATICS 31

FINAL TEST

GENERAL INSTRUCTIONS

YOU HAVE **THREE** HOURS TO COMPLETE THIS TEST. Work through the entire test answering the questions you are sure you know. You will then be able to concentrate on the questions of which you are not quite sure.

TOTAL MARKS: 100

PART A: Multiple Choice	80 marks
PART B: Written Response	20 marks

You may use any hand-held calculator. Calculators having graphing capabilities, built-in formulas, mathematical functions, or other programmable features are allowed.

Value

PART A: MULTIPLE CHOICE

80 Answer all multiple-choice questions on the Part A Response Page included in your test.

Read each question carefully and decide which of the choices BEST completes the statement or answers the question. Locate the question number on the Response Page and place your answer in the corresponding blank.

1. The solution, in interval notation, to the inequality $|2-x|<3$ is

A. $(-1,5)$
B. $(-5,-1)$
C. $(1,5)$
D. $(-5,1)$

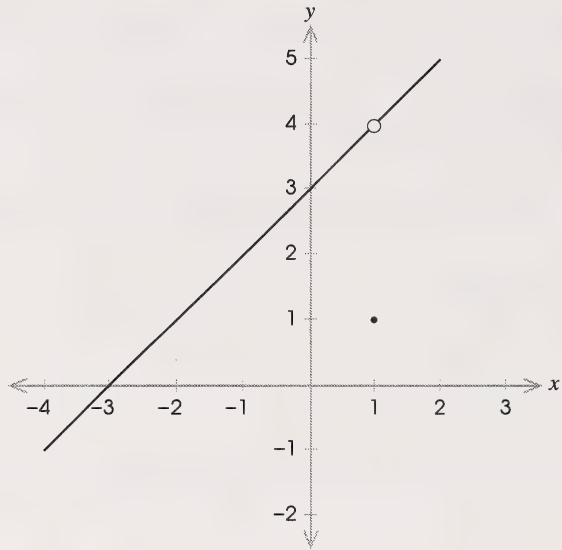
2. If $f(x)=\sqrt{1-x}$ and $g(x)=\sqrt{x+1}$, then the domain of the composition function $f \circ g$ is

A. $[0,1]$
B. $(-\infty,1)$
C. $(-1,\infty]$
D. $[-1,0)$

3. The graph of $y=f(x)$ is symmetric with respect to the y-axis if, for any value of x,

A. $f(-x)=-f(x)$
B. $f(-x)=f(x)$
C. $x=f(y)$
D. $-x=f(-y)$

Use the following information to answer question 4.



4. In the diagram, $f(x)=x+3$, when $x\neq 1$, and $f(x)=1$, when $x=1$. The value of $\lim_{x\to 1^-} f(x)$ is

A. 1
B. 2
C. 4
D. undefined

5. The limit of the sum $-\pi+\pi-\pi+\pi-\pi+\ldots$, as the number of terms increases without bound, is

A. $-\pi$
B. 0
C. π
D. undefined

6. If $f(x)=x-x^2$ and $g(x)=2x^2-x+3$, then $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ is

A. -2
B. $-\frac{1}{2}$
C. 0
D. $-\infty$

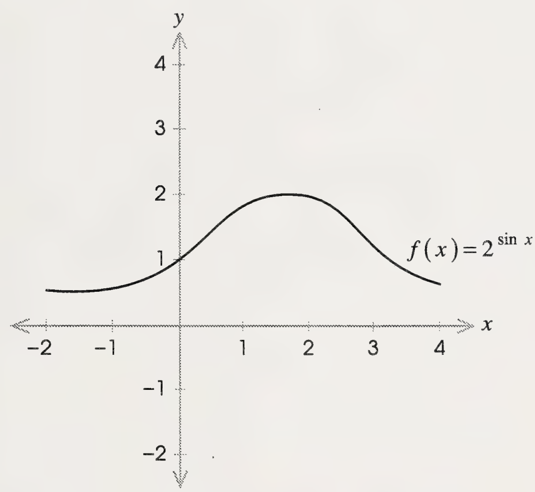
7. The equation of the secant through $f(x) = 2x^3$ at $x = 0$ and $x = -1$ is

A. $2x - y = 0$
B. $2x + y = 0$
C. $x - 2y = 0$
D. $x + 2y = 0$

8. The point on the graph of $y = (x+1)^{\frac{1}{3}}$ where the tangent is parallel to $x - 3y = 6$ is

A. $(-9, -2)$
B. $(-2, -1)$
C. $(-1, 0)$
D. $(0, 1)$

Use the following information to answer question 9.



9. The slope of $f(x) = 2^{\sin x}$ may be approximated by $\frac{f(x+h) - f(x)}{h}$.

Using $h = 0.001$, the approximation for the slope of the curve at $x = 0$, to the nearest hundredth, is

A. 0.68
B. 0.69
C. 0.70
D. 0.71

10. The slope of the tangent to the graph of the relation $xy^2 + y = 3$ at $(2, 1)$ is

A. $-\frac{1}{5}$

B. $\frac{2}{5}$

C. $-\frac{1}{2}$

D. 2

11. The derivative of $y = \frac{(4x-3)^3}{(3x-1)^2}$, in factored form, is

A. $2(4x-3)^2(3x-1)^{-1}$
B. $-6(4x-3)^2(3x-1)^{-1}$
C. $(x-9)(3x-1)^{-3}(4x-3)^2$
D. $6(2x+1)(3x-1)^{-3}(4x-3)^2$

12. Let function *f* be defined on the interval $[0, 3]$ as follows:

$f(x) = 4x$, where $0 \le x \le 1$
$f(x) = ax^2 + bx + 5$, where $1 < x \le 3$

If *f* is continuous and differentiable at $x = 1$, find a and b.

A. $a = -5,\ b = 4$
B. $a = 5,\ b = -6$
C. $a = 3,\ b = -4$
D. $a = -3,\ b = 2$

13. If $\cos\alpha = \frac{2}{5}$, then $\cos 2\alpha$ equals

A. $-\frac{4}{5}$

B. $\frac{4}{5}$

C. $-\frac{17}{25}$

D. $\frac{17}{25}$

14. The derivative of $y = \tan\left(x + \frac{\pi}{4}\right)$ is the same as the derivative of

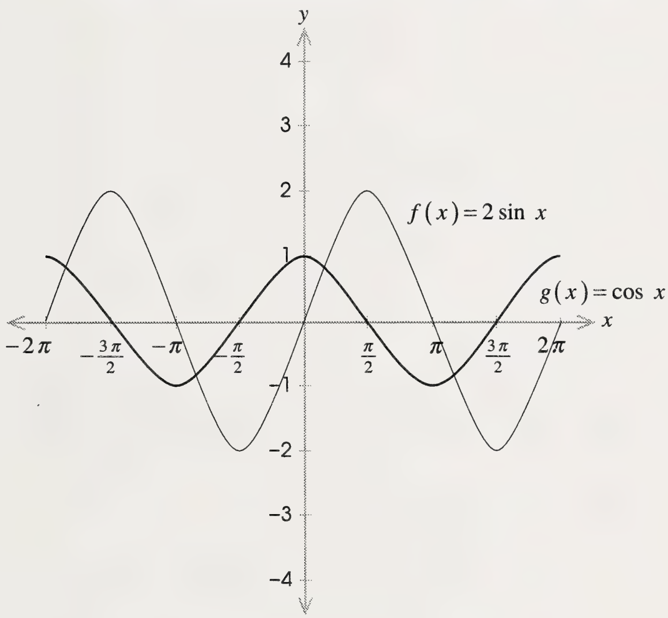
A. $y = \dfrac{2\tan x}{1 - \tan x}$

B. $y = \dfrac{2\tan x}{1 + \tan x}$

C. $y = \dfrac{1 + \tan x}{1 - \tan x}$

D. $y = \dfrac{1 - \tan x}{1 + \tan x}$

Use the following information to answer question 15.



15. If $f(x) = 2\sin x$ and $g(x) = \cos x$, then the solutions of the trigonometric equation $2\sin x = \cos x$ are

A. the y-intercepts of both f and g
B. the x-intercepts of both f and g
C. the y-values at the points of intersection of the graphs of f and g
D. the x-values at the points of intersection of the graphs of f and g

16. The derivative of $y = \cos^2 x$ is

A. $\cos 2x$
B. $-\cos 2x$
C. $\sin 2x$
D. $-\sin 2x$

17. The value of $\lim\limits_{x\to 0} \frac{\sin 3x}{4x}$ is

A. 0
B. 1
C. $\frac{3}{4}$
D. $\frac{4}{3}$

18. The oblique asymptote of $y = \frac{x^2+x}{x-1}$ is

A. $y = x+2$
B. $y = x-2$
C. $y = x+1$
D. $y = x-1$

19. If at point $P(a,b)$ on the graph of $y = f(x)$, the derivative of the function changes from positive values for $x < a$ to negative values for $x > a$, then

A. $x = a$ is a local maximum
B. $y = b$ is a local maximum
C. $x = a$ is a local minimum
D. $y = b$ is a local minimum

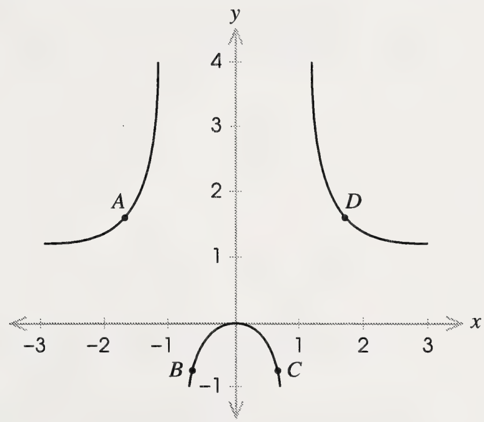
20. The interval where the function $y = \frac{x^3}{3} - 4x + 5$ is decreasing is

A. $(-\infty, 2) \cup (2, \infty)$
B. $(-2, 2)$
C. $(-\infty, 2\sqrt{3}) \cup (2\sqrt{3}, \infty)$
D. $(-2\sqrt{3}, 2\sqrt{3})$

21. A relative maximum of the function $f(x) = \frac{4}{x} + x$ is

A. -4
B. 2
C. -2
D. 4

Use the following information to answer question 22.



22. A point on the curve $y = f(x)$, where $f'(x) < 0$ and $f''(x) > 0$ is

A. A
B. B
C. C
D. D

23. If $f'(x) = (x-1)^2 (x+2)^3$, then the function f has

A. a local maximum at $x = 1$
B. a local minimum at $x = 1$
C. a local maximum at $x = -2$
D. a local minimum at $x = -2$

24. The curve $y = x\sqrt{1-x}$ is concave downward on the interval

A. $(-\infty, 1)$
B. $(-\infty, 1]$
C. $\left(\frac{2}{3}, 1\right)$
D. $\left(\frac{2}{3}, 1\right]$

25. The points of inflection of the graph of $y = \sin x + x$ occur at

A. $\pi + 2n\pi$, where $n \in I$
B. $\frac{\pi}{2} + n\pi$, where $n \in I$
C. $n\pi$, where $n \in I$
D. $\frac{\pi}{4} + 2n\pi$, where $n \in I$

Use the following information to answer questions 26 and 27.

An open box is constructed from a rectangular sheet of cardboard by cutting out squares from each corner and folding up the sides. The original sheet is 8 cm by 5 cm, and the squares have a side of *x* cm.

26. The volume *V* expressed as a function of *x* is

A. $V = 4x^3 - 26x^2 + 40x$
B. $V = 4x^3 - 13x^2 + 40x$
C. $V = 2x^3 - 26x^2 + 20x$
D. $V = 2x^3 - 13x^2 + 20x$

27. The maximum volume of the box is

A. 9 cm^3
B. 18 cm^3
C. 27 cm^3
D. 36 cm^3

28. A ladder, 5 m long, is resting against a gymnasium wall. The top of the ladder begins to slip down the wall at 1 m/s. How fast is the foot of the ladder moving away from the wall, when the top of the ladder is 3 m above the ground?

A. 1.33 m/s
B. 0.25 m/s
C. 0.75 m/s
D. 0.67 m/s

29. The position s at time t of an object travelling in a straight line is given by $s = 3t^3 - t^2 + 1$. What is the velocity of this object when its acceleration is 16?

A. 9
B. 7
C. 5
D. 3

30. The value of $\int_0^1 x\left(x^2+1\right)^2\, dx$ is

A. $\frac{2}{3}$

B. $\frac{4}{3}$

C. $\frac{7}{3}$

D. $\frac{7}{6}$

31. The equation of the curve whose slope at point (x, y) is $3x^2 - 2x$ and which passes through $(2, 1)$ is

A. $y = x^3 - x^2 - 3$
B. $y = 3x^2 - 2x - 7$
C. $y = 3x^3 - 2x^2 - 15$
D. $y = 6x - 11$

32. The solution of the differential equation $\frac{dy}{dx} = (\sin x \ \cot x)\ dx$ is

A. $y = -\sin x + C$
B. $y = \sin x + C$
C. $y = \cos x + C$
D. $y = -\cos x + C$

33. The area bounded by $y = x^2 - 2x$ and $y = x + 4$ is given by

A. $\int_{-1}^{4} \left(x^2 - 3x - 4\right)\ dx$
B. $\int_{-4}^{1} \left(x^2 - 3x - 4\right)\ dx$
C. $\int_{-1}^{4} \left(4 + 3x - x^2\right)\ dx$
D. $\int_{-4}^{1} \left(4 + 3x - x^2\right)\ dx$

34. The area bounded by the graph of $y = \sin x$ and the x-axis, from $x = 0$ and $x = \frac{\pi}{3}$ is

A. $\frac{3}{2}$
B. $\frac{1}{2}$
C. $\frac{2-\sqrt{3}}{2}$
D. $\frac{2+\sqrt{3}}{2}$

35. The acceleration a of an object moving in a straight line, at time $t \geq 0$ s, is given by $a = 2t + 3$. If its initial velocity is 5 m/s and its initial position is -7 m, what is its position at $t = 1$ s?

A. $-\frac{1}{3}$
B. $-\frac{1}{6}$
C. $\frac{1}{6}$
D. $\frac{1}{3}$

36. The mean value of $y = \sec^2 x$ over the interval $\left[0, \frac{\pi}{4}\right]$ is

A. $\dfrac{\pi}{4}$
B. $\dfrac{4}{\pi}$
C. 2
D. $\dfrac{1}{2}$

37. The equation of the normal to the curve $y = \ln x$ at $x = 1$ is

A. $x + y = 0$
B. $x - y = 0$
C. $x + y - 1 = 0$
D. $x - y + 1 = 0$

38. The slope of $y = 3^x$ at $x = 2$ is

A. $\dfrac{9}{\ln 3}$
B. $9 + \ln 3$
C. $9 - \ln 3$
D. $9 \ln 3$

39. The area between the graph of $y = \frac{1}{x}$ and the x-axis from $x = 2$ to $x = 10$ is

A. $\dfrac{\ln 10}{\ln 2}$
B. $\ln(10 - 2)$
C. $2 \ln 5$
D. $\ln 5$

40. If $\frac{dy}{dx} = ky$, $y = 2$ when $x = 0$, and $y = 4$ when $x = 1$, then the value of k is

A. $\dfrac{\ln 4}{2}$
B. $\dfrac{\ln 4}{\ln 2}$
C. $\ln 2$
D. $\ln 4$

PART A: RESPONSE PAGE

_______ 1.	_______ 16.	_______ 31.
_______ 2.	_______ 17.	_______ 32.
_______ 3.	_______ 18.	_______ 33.
_______ 4.	_______ 19.	_______ 34.
_______ 5.	_______ 20.	_______ 35.
_______ 6.	_______ 21.	_______ 36.
_______ 7.	_______ 22.	_______ 37.
_______ 8.	_______ 23.	_______ 38.
_______ 9.	_______ 24.	_______ 39.
_______ 10.	_______ 25.	_______ 40.
_______ 11.	_______ 26.	
_______ 12.	_______ 27.	
_______ 13.	_______ 28.	
_______ 14.	_______ 29.	
_______ 15.	_______ 30.	

Name of Student ______________________ Student I.D. # ______________

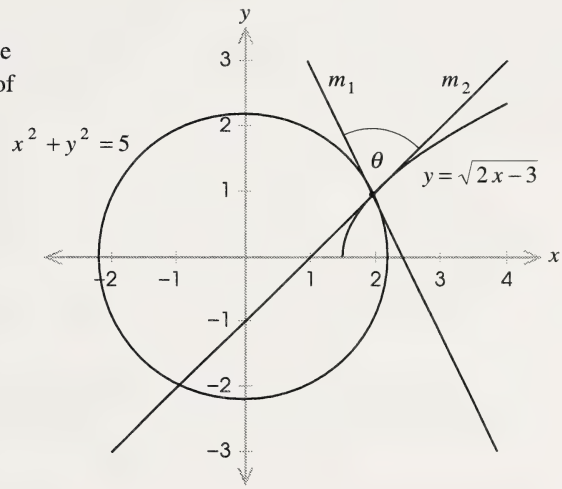
Name of School ______________________ Date ______________

Value

PART B: WRITTEN RESPONSE

20 **Answer each of the following questions using the space provided.**

1. In the diagram, the graphs of $x^2 + y^2 = 5$ and $y = \sqrt{2x - 3}$, and the tangents to those curves at their point of intersection are shown.



(3 marks) **a.** Determine the coordinates of the point of intersection of $x^2 + y^2 = 5$ and $y = \sqrt{2x - 3}$.

Name of Student ______________________	Student I.D. # ______________
Name of School ______________________	Date ______________

(4 marks)

b. Using derivatives, find the slopes of the tangent lines m_1 and m_2, drawn to $x^2 + y^2 = 5$ and $y = \sqrt{2x-3}$ at their point of intersection.

(1 mark)

c. The angle θ between the two tangents can be determined from their slopes using $\tan\,\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$. Determine the angle between the tangent lines, as shown in the diagram.

Name of Student		Student I.D. #	
Name of School		Date	

(3 marks) 2. Explain why the definite integral from 0 to 2π of the function $f(x) = 2 \sin x$ is zero, whereas the definite integral from 0 to π of the same function is nonzero. Use a sketch to support your answer.

(4 marks) 3. Use the definition of the derivative to determine the first derivative of $y = \sqrt{3 - 5x}$.
Note: No marks will be awarded for derivations that use any of the derivative theorems.

Name of Student ______________________	Student I.D. # ______________
Name of School ______________________	Date ______________

(5 marks) 4. Determine the area between the curve $y = x^2 + 3x - 28$ and the x-axis, from $x = -8$ to $x = 0$.

Name of Student	____________________	Student I.D. #	________________
Name of School	____________________	Date	________________

TEACHER QUESTIONNAIRE FOR MATHEMATATICS 31

This is a course designed in a new distance-learning format, so we are interested in your responses. Your constructive comments will be greatly appreciated so that a future revision may incorporate any necessary improvements.

Teacher's Name ______________________ Area of Expertise ______________________

School Name ______________________ Date ______________________

Design

1. The modules follow a definite systematic design. Did you find it easy to follow?

   ❑ Yes ❑ No If no, explain.

   ____

   ____

2. Did your observations reveal that the students found the design easy to follow?

   ❑ Yes ❑ No If no, explain.

   ____

   ____

3. Did you find the Learning Facilitator's Manual helpful?

   ❑ Yes ❑ No If no, explain.

   ____

   ____

4. Part of the design involves stating the objectives in student terms. Do you feel this helped the students understand what they were going to learn?

   ❑ Yes ❑ No If no, explain.

   ____

   ____

5. The Learning Facilitator's Manual contains Assignment answers and a sample test. Did you find these helpful?

❑ Yes ❑ No If no, explain.

____

____

6. Did the Follow-up Activities prove to be helpful?

❑ Yes ❑ No If no, explain.

____

____

7. Were students motivated to try these Follow-up Activities?

❑ Yes ❑ No If no, give details.

____

____

8. Suggestions for computer and video activities are included in the course. Were your students able to use these activities?

❑ Yes ❑ No Comment on the lines below.

____

____

9. Were the assignments appropriate?

❑ Yes ❑ No If no, give details.

____

____

10. Did you fax assignments? ❑ Yes ❑ No

11. If you did fax, did you get satisfactory results from using this procedure?

❑ Yes ❑ No If no, give details.

____

____

Instruction

1. Did you find the instruction clear?

   ❑ Yes ❑ No If no, give details.

2. Did your observations reveal that the students found the instruction interesting?

   ❑ Yes ❑ No If no, give details.

3. Did you find the instruction adequate?

   ❑ Yes ❑ No If no, give details.

4. Was the reading level appropriate?

   ❑ Yes ❑ No If no, give details.

5. Was the work load adequate?

   ❑ Yes ❑ No If no, give details.

6. Was the content accurate and current?

   ❑ Yes ❑ No If no, give details.

7. Did the content flow consistently and logically?

❑ Yes ❑ No If no, give details.

8. Was the transition between booklets smooth?

❑ Yes ❑ No If no, give details.

9. Was the transition between print and other media smooth?

❑ Yes ❑ No If no, give details.

Additional Comments

Thanks for taking the time to complete this survey. Your feedback is important to us.

Fax Number: 674-6686

Instructional Design and Development Unit
Alberta Distance Learning Centre
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Note: Please ensure that each of your students has completed and forwarded a copy of the Course Survey.

Mathematics 31

Learning Facilitator's Manual

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